PART-I

SKM ACADEMY

REAL ANALYSIS

NET/GATE/JAM

Name

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Subject

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SETS

SET :- The collection of well defined objects is called set.

Different Authors says Different About set.

1) A Set is a well defined collection of Distinct objects.

2) A set is well defined Collection of objects.

3) A set is collection of distinct objects.

4) A Set is collection of objects.

5) A set is Not Defined.

6) A set is a collection of unordered objects.

7) A set is a well defined collection of distinct objects as our thought.

i.e, we can say collection of all such type of elements or objects which satisfy some Rule. It is possible to say, whether a Particular objects belongs to the collection or Not.

Equal set :- Two sets are said to be equal when they consists of exactly the same elements.

i.e Both have same Number of elements and cardinality is also same.
i.e $A = B$ iff $A \subseteq B$ or $B \subseteq A$ and $|A| = |B|$

Cardinality of a Set :- The Number of elements in any set A is called cardinality of A. It is denoted by card(A) or $|A|$

E.g = $S = \{a, e, i, o, u\}$
$|S| = 5$

Subset :- if S and T are two sets set each member of S is member of T Then S is subset of T denoted as $S \subseteq T$.

i.e $x \in S \Rightarrow x \in T$
$\Rightarrow S \subseteq T$

Note: if $x \in A \Rightarrow x \in B$ and if there is $y \in B$ but $y \notin A$, then A is called Proper Subset of B i.e $A \subset B$.

Note: Every set is subset of itself.
Note: Empty set is a subset of every set.

Note: Empty set is unique.

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BOUNDED SETS IN ORDERED FIELDS.

Def:- Suppose that F is an ordered field, $A \subseteq F$, and $u \in F$. We say that:

1) u is an upper bound for A if $\forall x \in A$, $x \le u$.
2) u is a lower bound for A if $\forall x \in A$, $x \ge u$.
3) u is a Maximum (or greatest) element for A if $u \in A$. and $\forall x \in A$, $x \le u$. i.e $u = \max A$.
4) u is a minimum (or least) element for A if $u \in A$ and $\forall x \in A$, $x \ge u$. i.e $u = \min A$.

Maximal element :- Let F be partially ordered set & $M \subseteq F$, we say $M \in F$ is called Maximal element if No member of F which contains M other than M itself.

Minimal element :- Let F be P.O.S, we say $S \in F$ is called Minimal element if No element of F which is contained in S other than S itself.

Thm. a) A set can have more than one upper bound and more than one lower bound.

b) A set cannot have more than one Maximum and more than one minimum element.

c) Every Non-Empty Finite set has both a Maximum element and a minimum element.

d) $A \neq \phi$, closed set and bdd above $\Rightarrow$ A has Maximal element

e) $A \neq \phi$, closed set and bdd below $\Rightarrow$ A has minimal element

Result: An infinite set, even when bounded, may or may not

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have a Maximum or a Minimum element. For Ex. "open interval" (a,b) has Neither a minimum element. Nor a Maximum element.

Ex :- $F = \{\{1,2\}, \{2\}, \{2,3\}, \{2,3,4\}, \{5\}\}$

It is P.O.S with Relation $\subseteq$

Relation :- A Relation on set A is called P.O. Partial ordered Relation if

(i) Reflexive

(ii) Anti symmetric

(iii) Transitive.

Partial ordered set :- A set is equipped with Partial ordered Relation is called Partial ordered Set.

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Maximal → $\{2,3,4\}, \{1,2\}, \{5\}$

Minimal → $\{2\}, \{5\}$

Maximal and minimal – $\{5\}$

$\{2,3\}$ Neither Maximal Nor Minimal.

$\{\mathbb{N}, \leq\}$:- 1 is minimal element & No Maximal element

$\{$ [illegible] $\}$:- No Maximal & No minimal element.

Bounded set :- Let S be a Subset of $\mathbb{R}$ $(S \subseteq \mathbb{R})$ then we say set S is Bdd if $\exists$ $m, M \in \mathbb{R}$ s.t

$$\boxed{m \leq x \leq M \quad \forall\ x \in S}$$

Here 'm' and 'M' are called lower Bound and Upper Bound Respectively.

Note :- if 'm' is known and M is unknown then S is called bdd below and unbdd above. Hence S is unbdd.

Eg :- $S = \mathbb{N} = \{1, 2, 3, - - - - -\}$

$m = 1, \quad M = \text{unknown}.$

Note :- if m is unknown and 'M' is known ~~then~~ S is called bdd above and unbdd Hence S is unbdd.

Eg :- $S = \{-1, -2, - - - - - - -\}$

$M = -1, \quad m = \text{unknown}.$

Note :- If Both 'm' , 'M' are unknown. then we say S is unbdd from both Side.

Eg :- $S = \mathbb{Z} = \{- - - - , -3, -2, -1, 0, 1, 2, 3, - - - - \}$

M unknown , m unknown.

Note:- Bounds of sets are infinite Number.

Supremum of a set :- The least upper Bound (L.U.B) of a set S, if exists, is called the supremum of that set.

i.e if the set of all upper Bounds of a set S has smallest Number say 'M', then M is called L.U.B or supremum of S.

Note:- Supremum of set S is very closely Related to the set S.

i.e if $\sup S = M$

$\Rightarrow \forall\ x \in S\ ;\ x \leq M$

for any $\epsilon > 0$ $\exists\ t \in S$ s.t $t > M - \epsilon$

i.e if we substract smaller and smaller value ϵ from $\sup S$ then there is one element in S which will greater to $M - \epsilon$

Infimum of a set :- The g.l.b of a set S if exists is called the infimum of that set.

i.e if the set of all lower Bounds of S has largest number say 'm', then m is called g.l.b or Infimum of set S.

Note:- Infimum of set S is very closely Related to set S i.e if $\operatorname{Inf} S = m \Rightarrow \forall\ x \in S,\ x \geq m$

for any $\epsilon > 0\ \exists\ t \in S$ s.t $t < m + \epsilon$

Note:- Sup and Inf of a set are always unique

Note:- Bounds of set may or may Not be belongs to the set, If they belongs to the set called largest and smallest element of that set.

Eg:- $S = \left\{ \frac{1}{n},\ n \in \mathbb{N} \right\}$

Lower bound $= 0 \notin S$

Upper bound $= 1 \in S$.

2) $S = \left\{ -\frac{(n+1)}{n},\ n \in \mathbb{N} \right\}$

$= \left\{ -2,\ -\frac{3}{2},\ -\frac{4}{3},\ -\frac{5}{4},\ \text{---}\ \text{---} \right\}$

$\sup S = -1 \notin S$, $\inf S = -2 \in S$

Results:-

1) Every Bounded Subset of $\mathbb{R}$ has sup and Inf in $\mathbb{R}$.

2) Every Subset of Bounded Set is Bounded.

3) Every finite set is Bounded.

Note:- Subset of an unbdd set is Not Necessarily unbdd.

Eg $[a, b] \subset \mathbb{R}$ and $\mathbb{R}$ is unbdd.

$[a, b]$ is bdd. if a and b are finite.

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Note:- Every infinite set Need Not be unbdd.

Eg:- $S = \{x \in \mathbb{R},\ 1 < x < 3\}$ is infinite but bdd by 1 and 3.

4> If 'u' is an upper bound of a set S and $u \in S$ then $u = \sup S$

5> If 'l' is an lower bound of a set S and 'l' $\in S$ then $l = \inf S$.

6> Every Non-empty Set of Real Numbers which is bdd above has the least upper bound.

7> Every Non-empty set of Real Numbers which is Bounded below has the greatest lower bound

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8> (i) $\sup(A+B) = \sup A + \sup B$

(ii) $\sup(A-B) = \sup A - \inf B$

(iii) $\inf(A+B) = \inf A + \inf B$

iv> $\inf(A-B) = \inf A - \sup B$

Here $A+B = \{a+b,\ a \in A,\ b \in B\}$

$A-B = \{a-b,\ a \in A,\ b \in B\}$.

v> $\sup(A \cup B) = \max\{\sup A, \sup B\}$

vi> $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

9> if A is bdd Subset of $\mathbb{R}$ then $\lambda . A$ is also bdd $(\lambda \in \mathbb{R})$ And.

*> Let X be a Nonempty Set and Suppose $f, g : X \to \mathbb{R}$ are functions whose ranges are bounded Sets of Real no.

[illegible]

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i) $Sup(\lambda \cdot A) = \lambda \cdot sup A$ if $\lambda > 0$

ii) $Inf(\lambda \cdot A) = \lambda \cdot Inf A$ if $\lambda > 0$

iii) $Sup(\lambda \cdot A) = \lambda \cdot Inf A$ if $\lambda < 0$

iv) $Inf(\lambda \cdot A) = \lambda \cdot sup A$ if $\lambda < 0$.

10) Let $A \neq \phi$ & $B \neq \phi$ Subset of +ve Real No. & bdd. above & write $AB = \{ab ; a \in A, b \in B\}$ $\Rightarrow$ AB is bdd above & $sup(AB) = Sup A \times Sup B$

Que Find the Sup. and Inf. of Set S.

1) $S = \{\sin\frac{1}{n}, n \in \mathbb{N}\}$

$= \{\sin 1, \sin\frac{1}{2}, \sin\frac{1}{3}, - - -\}$

$\because$ $\frac{1}{n}$ is lies b/w 0 and 1 So $\sin\frac{1}{n}$ will given its sup. and inf. when $n=1$ and at 0 when $n \to \infty$

Hence, $sup S = \sin 1 \in S$ and $Inf S = \sin 0 \notin S$ $\left\{ \sin\frac{1}{n} \text{ is } \downarrow \text{ on } (0,1) \right.$.

2) $S = \{\cos\frac{1}{n}, n \in \mathbb{N}\}$

$Sup S = \cos 0 = 1 \notin S$

$Inf S = \cos 1 \in S$.

3) $S = \{x \in \mathbb{R} ; x = (-1)^n \cdot n, n \in \mathbb{N}\}$

$= \{ - - - , -3, -1, 2, 4, - - - \}$

S is unbdd from both side. Hence sup S & Inf S doesn't exist.

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Archemedean Property :-

i) For any $\delta > 0$ $\exists$ $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$ $\forall$ $n > m$

ii) Let $\forall$ $x, y \in \mathbb{R}$ and $y > 0$ $\exists$ $n \in \mathbb{N}$ s.t. $x \leq ny$

Interval :- The set of all points b/w 'a' and 'b' is called an interval where $a, b \in \mathbb{R}$. i.e let $S \subseteq \mathbb{R}$, we say S is interval, whenever $a, b \in \mathbb{R}$ and $a < x < b \Rightarrow x \in S$.

I) Closed Interval :- $S = \{x : a \leq x \leq b\}$
i.e $S = [a, b]$

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II) open Interval :- $S = \{x : a < x < b\}$

III) Semi open or Semi closed interval :-

$S = \{x : a < x \leq b\}$ or $\{x : a \leq x < b\}$
i.e $S = (a, b]$ or $S = [a, b)$

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Note :- Let S be an interval then $\sup S - \operatorname{Inf} S$ gives the length of interval

Note :- If length is finite then called finite interval otherwise infinite.

$[0, 2]$ is finite interval because $|0 - 2| = 2$.

$[0, \infty)$ is infinite interval length is not finite.

Result :- Every bdd set is subset of an interval of closed finite length.

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Power set :- The set of all subsets of any set S is said to be a Power set, and it is denoted by $P(S)$.

E.g :- $S = \{p, q, r\}$

Then $P(S) = ?$

$P(S) = \{\phi, S, \{p\}, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}\}$

$|P(S)| = 2^{|S|} = 2^3 = 8.$

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Note * $\boxed{|A| = n \text{ then } |P(A)| = 2^n.}$

* ~~Card~~ **Cartesion Product of sets :-**

For any two sets A and B, Cartesion Product of A and B by $A \times B$ and defined as

$A \times B = \{(a,b) ; a \in A, b \in B\}$

$A \times B \times C = \{(a,b,c) ; a \in A, b \in B, c \in C\}$.

In general

$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n), a_i \in A \; \forall \; i = 1 \text{ to } n\}$

Here $(a_1, a_2, \ldots, a_n)$ called an ordered n-tupple

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Results:-

1) if $|A| = m$ & $|B| = n$ then
$|B \times A| = |A \times B| = m \times n = m \cdot n$.

2) $A \times B \Longleftrightarrow B \times A$ iff $A = B$

3) if $|A \cap B| = m$ then $|(A \times B) \cap (B \times A)| = m^2$

4) $A \subset B \Rightarrow A \times C \subset B \times C$

5) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

6) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

7) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

8) $(A \times C) - (B \times C) = (A - B) \times C$

9) $A \subset B$ and $C \subset D$ then $A \times C \subset B \times D$

Union and Intersection of sets!

$A \cup B = \{x ; x \in A \text{ or } x \in B\}$

$A \cap B = \{x ; x \in A \text{ and } x \in B\}$

Properties :- For any sets A, B, and C.

1) $A \subset A \cup B$

2) $B \subset A \cup B$

3) $A \cup A = A$ (Idempotent law)

4) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative law)

5) $A \cup A \cap B \subset A$

6) $A \cap B \subset B$

7) $A \cap \phi = \phi$

8) $A \cap A = A$ (Intersection is Idempotent)

9) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative)

Disjoint set :- Two sets A and B are Said to be Disjoint if they have No element in common, means there intersection should be ϕ.

i'e $\boxed{A \text{ and } B \text{ are disjoint if } A \cap B = \phi}$

Note :- Every Disjoint sets are distinct sets, Not Conversely.

Eg $S = \{1, 2, 3, 4\}$, $T = \{2, 5, 7, 8\}$.
Here S and T are distinct but Not disjoint

$S \cap T = \{2\} \neq \phi$.

Difference of two sets :-

$A - B = \{x : x \in A \text{ and } x \notin B\}$

All elements which are in A but Not in B

Note: 1. Difference Not Commutative
i'e $A - B \neq B - A$.

Note - 2 Difference Not associative.
i'e $(A - (B - C) \neq (A - B) - C$

Symmetric Difference :- Let X and Y are two sets then Symmetric Difference of sets X and Y denoted by $X \Delta Y$ and defined as

$X \Delta Y = (X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$

Equivalent / Similar set :- Two sets A and B are said to be equivalent if ∃ a bijective map from A to B.

Or Two finite sets are said to be similar/ equivalent sets if they have same cardinality.

Eg. -
$A = \{a, b, c\}$, $|A| = 3$
$B = \{Ram, Shyam, Sita\}$, $|B| = 3$
$C = \{1, 2, 3\}$, $|C| = 3$
$D = \{2, 1, 3\}$, $|D| = 3$
$E = \{1, 2, 3, 4\}$, $|E| = 4$.

Here C and D are equal sets.
A, B, C, D similar/ equivalent sets.

Note :- if $|A| \neq |B|$ then A can't be similar to B.

Note :- if sets A and B are infinite then we can't denote their cardinality Numerically. So Here we introduce a New "Transfinite Number" to denote the cardinality of infinite set.

Transfinite Number :- For the infinite sets, we adopt a symbol to denote the cardinality of that set. These symbols are called transfinite Number or Cardinal Number.

Note : $\mathbb{N}$, Set of Natural Number (Infinite Set), Here we adopt a cardinal Number $\aleph_0$ (Aleph Naught) i.e. $|\mathbb{N}| = card(\mathbb{N}) = \aleph_0$

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Note $\aleph_0$ is the smallest transfinite Number. As 1 is smallest counting Number, 2 is second and so on. Similarly $\aleph_0$ is smallest transfinite Number and second transfinite Number is C (continum Hypothesis)

E.g $|\mathbb{N}| = \aleph_0$ & $|\mathbb{R}| = C$

$$|\mathbb{N}| < |\mathbb{R}| \Rightarrow \aleph_0 < C$$

Note:- Two infinite sets are similar iff they have the same transfinite Number.

E.g. $\mathbb{N}$ is Not similar to $\mathbb{R}$ because $\aleph_0 \neq C$

# A finite set can't be similar to it's Proper Subse

Note- Similarity hold the equivalence Relation.

PF:- Let A, B, C are 3 sets

1) Reflexive $\rightarrow A \sim A \ \forall$ sets.
2) Symmetric $\rightarrow A \sim B \Rightarrow B \sim A$
3) Transitive $\rightarrow A \sim B$ and $B \sim C$
$\Rightarrow A \sim C$.

Note: Set of Natural Number is similar to set of integer (Both are infinite set and have the same transfinite cardinal Number).

Pf. For this define map $f: \mathbb{N} \rightarrow \mathbb{Z}$

$$f(n) = (-1)^{n} \left[\frac{n}{2}\right]$$

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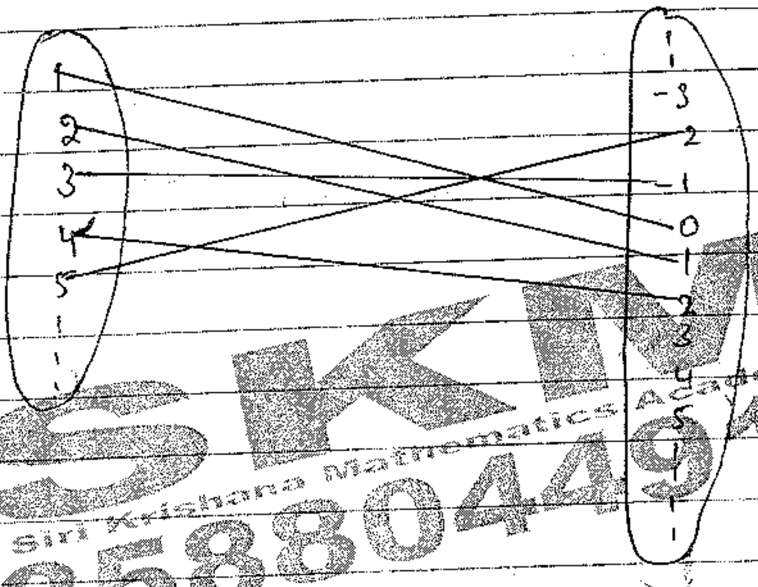

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So each element has distinct image and each element of co-domain has a pre-Image.

$\Rightarrow$ Map is one-one and onto $\Rightarrow$ Bijective Map.

$\therefore N \sim \mathbb{Z}$.

Eg-1 $f : \mathbb{N} \longrightarrow B_1 = \{1^2, 2^2, 3^2, \ldots\}$

then $f(n) = n^2$ is Bijective map

$\Rightarrow N \sim B_1$

Eg-2 $f : \mathbb{N} \longrightarrow B_2 = \{1/2, 2/3, 3/4, \ldots\}$

then $f(n) = \frac{n}{n+1}$ is Bijective map.

$\Rightarrow N \sim B_2$

Eg-3 $\mathbb{N} \longrightarrow C = \{2, 4, 8, 16, \ldots\}$

then $f(n) = 2^n$ is bijective map.

$\Rightarrow \mathbb{N} \sim C$

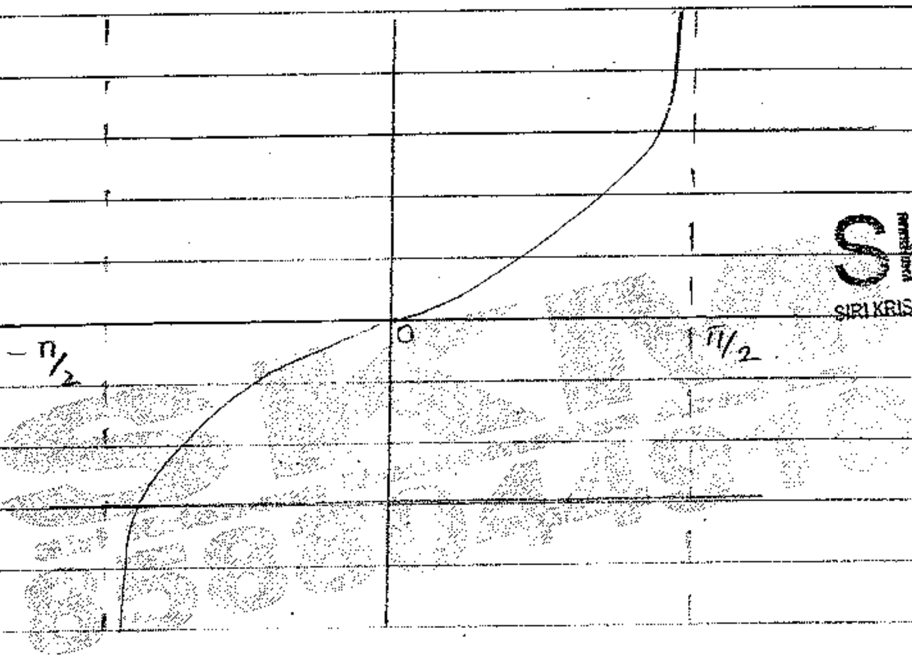
Here B and C are proper subsets of $\mathbb{N}$ and are similar to $\mathbb{N}$.

Note:- Infinite set can be similar to it's proper subset

Ex:- $A = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $B = \mathbb{R} = (-\infty, \infty)$

Define $f : (-\pi/2, \pi/2) \longrightarrow (-\infty, \infty)$

$f(x) = \tan x$.



Every point in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ have a unique image, or we say that every line $\parallel$ to x-axis cuts at exactly one point implies map is bijective.

$\Rightarrow A \sim B$

i.e. $(-\pi/2, \pi/2) \sim \mathbb{R}$.

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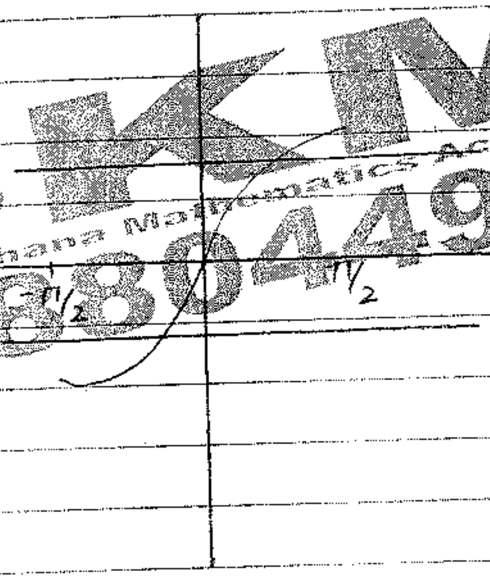
2) $A = [-1, 1]$, $B = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Define $f : B \longrightarrow A$

$f(x) = \sin x$.



cut exactly one point.

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Again, Every point in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has unique image, or, every line $\parallel$ to x-axis cuts the curve at exactly one point implies map is bijective, So $A \sim B$.

i.e $[-1, 1] \sim \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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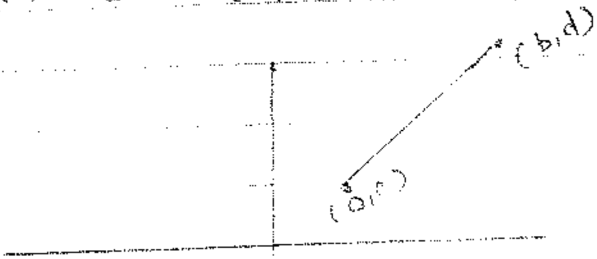
3) For Two sets $[a, b]$ and $[c, d]$ to be similar, we have,

Let (a, c) and (b, d) be two points in plane.



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By distance formula.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - c = \frac{d-c}{b-a}(x-a)$$

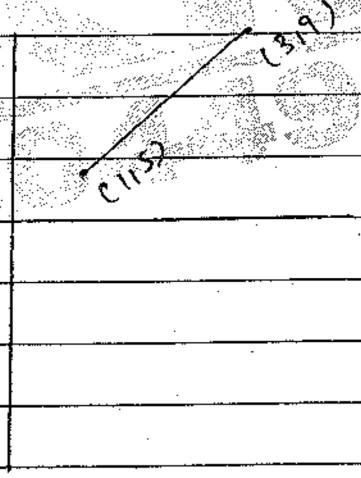
$$\Rightarrow \boxed{y = \frac{d-c}{b-a}(x-a) + c}$$

So, $f : [a,b] \to [c,d]$ given by

$f(x) = \frac{d-c}{b-a}(x-a) + c$ will given Bijective map.

So $[a,b] \sim [c,d]$.

Eg:- $A = [a,b] = [1,3]$, $B = [c,d] = [5,9]$.



Define $f : A \to B$
by $f(x) = \frac{9-5}{3-1}(x-1) + 5$

$f(x) = 2(x-1) + 5$

This is linear function f it is always one-one and

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onto from A to B

$\Rightarrow A \sim B$

i.e $[1,3] \sim [5,9]$.

Check:

Put $x = 1$

$f(1) = 2(1-1)+5 = 5$

So, $1 \rightarrow 5$ true.

Put $x = 3$

$f(x) = 2(3-1)+5 = 9$

So, $3 \rightarrow 9$ true.

ex4 $A = [-1,1]$, $B = [2,6]$.

Define $f : A \rightarrow B$

$$f(x) = \frac{6-2}{1+1}(x+1)+2$$

$$f(x) = 2(x+1)+2$$

So $A \sim B$

Check: Put $x = -1$

$f(x) = 2(0)+2 = 2$

So $-1 \rightarrow 2$ True.

Put $x = 1$; $f(x) = 4+2 = 6$

So, $1 \rightarrow 6$ True.

Result: Any two closed or open interval are always equivalent.

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Countable

Countable set
- Empty
- Finite
- Denumerable or Enumerable or Countably infinite

Countable set :- Any set 'A' is said to be countable if it is finite or countably infinite.

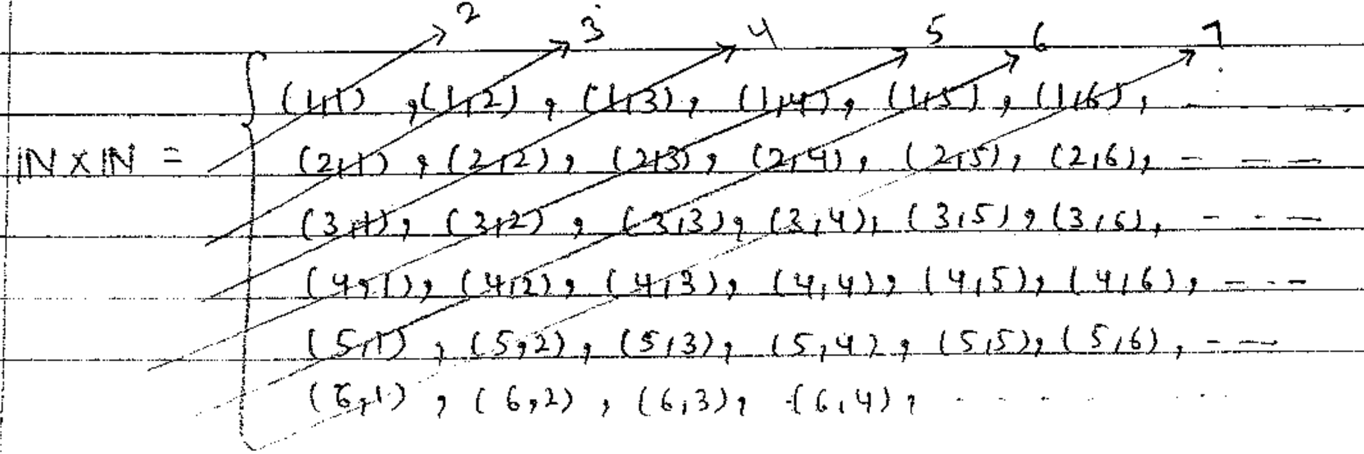
Countably infinite :- An Infinite set A is said to be countably Infinite if $\exists$ a bijective map from $\mathbb{N}$ to A or
A is said to be countably infinite if $A \sim N$.

And which is Not countable, called uncountable.

Note :- Term countable means, count the elements.

i.e if we level the elements of the set then it should be treated as $\{a_1, a_2, \text{---} a_n, a_{n+1}, \}$.

Eg:- $\mathbb{N} \times \mathbb{N}$ is c'ble.



$$\mathbb{N}\times\mathbb{N} = \left\{\begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \text{---} \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \text{---} \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \text{---} \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \text{---} \\ (6,1), (6,2), (6,3), (6,4), \cdots \end{array}\right.$$

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Here, we can level the elements of $\mathbb{N}\times\mathbb{N}$ by placing the elements as, first put those elements whose component sum is 2, after put those elements whose component sum is 3, then put for sum 4. similarily we can level the elements of $\mathbb{N}\times\mathbb{N}$

Here $\mathbb{N}\times\mathbb{N}$ is c'ble.

Ex:- 1) $\mathbb{N}$ is similar to set of integer.

$f: \mathbb{N} \longrightarrow \mathbb{Z}$

$$f(n) = \begin{cases} -\frac{n}{2}, & n \text{ is even} \\ \frac{n-1}{2}, & n \text{ is odd} \end{cases}$$

$\Rightarrow$ f is 1-1 and onto, Hence $\mathbb{N} \sim \mathbb{Z}$.

Ex-2> $f: \mathbb{Q}^+ \longrightarrow \mathbb{N}\times\mathbb{N}$

s.t $f\left(\frac{p}{q}\right) = (p,q)$

f is one to one

$\Rightarrow \mathbb{Q}^+ \sim \mathbb{N}\times\mathbb{N}$

And $\mathbb{N}\times\mathbb{N}$ is countable

$\Rightarrow \mathbb{Q}^+$ is countable.

Ex→3 $f: \mathbb{N} \longrightarrow 2\mathbb{Z}^+$ (Set of +ve Even Integer)

s.t $f(n) = 2n$.

f is 1-1 & onto.

$\Rightarrow \mathbb{N} \sim 2\mathbb{Z}^+$

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Note:- $\mathbb{N} \sim \mathbb{Q}^+ \sim \mathbb{Z} \sim \mathbb{N}\times\mathbb{N} \sim \mathbb{Q} \sim \mathbb{Q}^- \sim \mathbb{Q}^+ \sim 2\mathbb{Z}$

i'e cardinality of two similar sets are always equal so.

$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N}\times\mathbb{N}| = |\mathbb{Q}| = |\mathbb{Q}^+| = |\mathbb{Q}^-| = |2\mathbb{Z}| = \aleph_0$

Note:- Every infinite set has countably infinite subset.

Note: If $A \to B$ is one-one then $|A| \leq |B|$ then there exist $X \subseteq B$ s.t $A \sim X$.

Note: if $f: A \to B$ is onto $\Rightarrow |A| \geq |B|$ then $\exists$ $Y \subseteq A$ s.t $B \sim Y$.

Results:- Every infinite set is similar to atleast one of its proper subset.

i'e X is an infinite set then $\exists$ $Y \subset X$ s.t $Y \sim X$.

i'e X is an infinite set and A be any finite subset of X then $X \sim (X - A)$.

Eg:- $[0,1] \sim (0,1)$

$X = [0,1]$, $A = \{0,1\}$, $A - X = (0,1)]$.

Here. $[0,1] \sim [0,1) \sim (0,1] \sim (0,1)$.

Result:- Every subset of countable set is countable.

A be any set. Then following conditions are equivalent:-

1) A is countable.
2) ∃ a one-one map from A to $\mathbb{N}$
3) ∃ a onto map from $\mathbb{N}$ to A.

Dedekind Theorem :- Let A be any set. Then following are equivalent :-

1) A is infinite.
2) A is equivalent to atleast one proper subset of itself.

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Galois Paradox :- Every c'bly infinite set A can be put in one-one correspondence with a proper subset of A.

$\mathbb{N}$ → countably infinite.
$\mathbb{Z}$ → countably infinite
$\mathbb{Q}$ → countably infinite
$\mathbb{Q}^c$ → unc'bly infinite
$\mathbb{R}$ → "
$[a,b]$ → "
$(a,b]$ → "
$[a,b)$ → "
$(-\infty,a)$ → "
(a,∞) → "
$[a,\infty)$ → "
$(-\infty,a]$ → "

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Uncountable infinite sets / Non-Enumerable sets / Non-Denumerable sets :-

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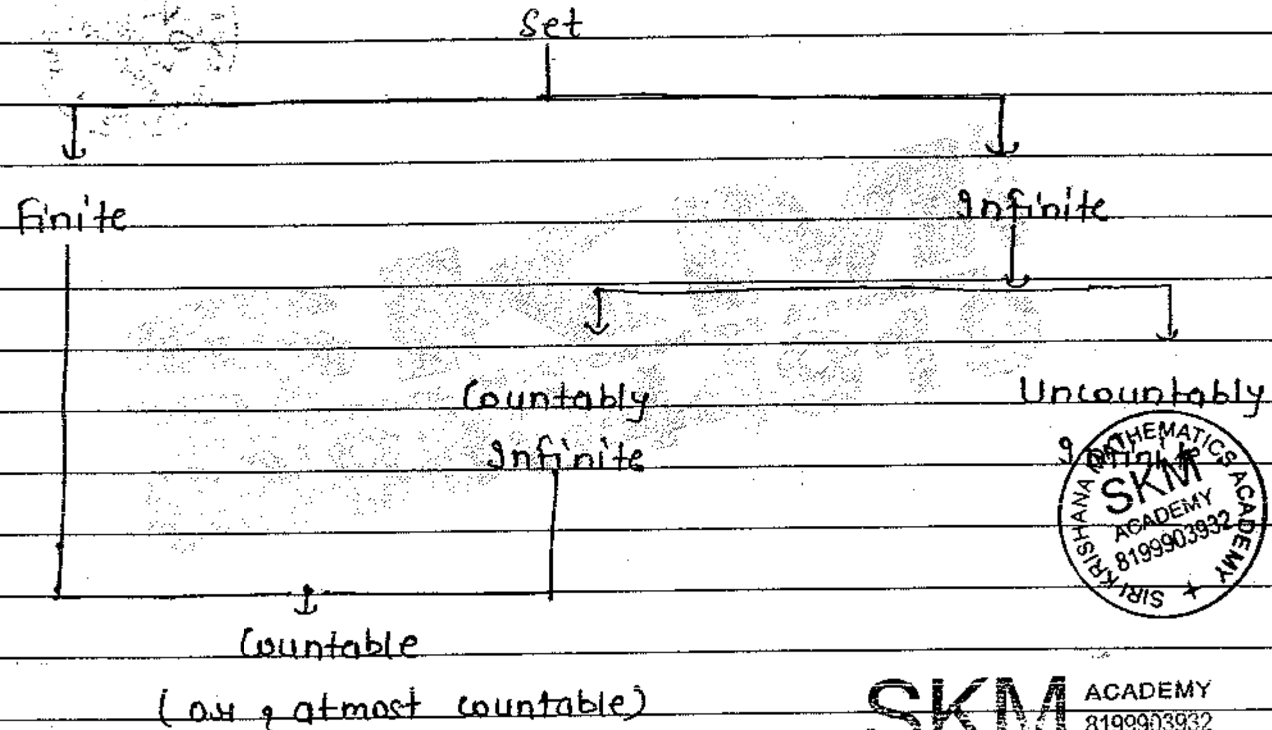
An infinite set A is said to be unc'ble infinite if ∃ No bijective map from $\mathbb{N}$ to A.


Set
Finite
Infinite
Countably Infinite
Uncountably Infinite
Countable
(or, atmost countable)

Results

1) A Subset of c'ble set is c'ble.
2) countable union of countable set is c'ble.
3) countable intersection of countable set is countable
4) finite Product of countable set is countable
5) A Set has unc'ble subset, then set is unc'ble.
6) Let A and B are two subset of Real Number and A is c'ble and B is unc'ble, then

a) $A-B$ is c'ble
b) $A \cap B$ is c'ble
c) $B-A$ is unc'ble
d) $A \cup B$ is unc'ble.

Eg:- $A = \mathbb{Q}$; $B = \mathbb{Q}^c$

1) $A-B = \mathbb{Q}-\mathbb{Q}^c = \mathbb{Q}$ → c'ble
2) $A\cap B = \mathbb{Q}\cap\mathbb{Q}^c = \phi$ → c'ble
3) $B-A = \mathbb{Q}^c-\mathbb{Q} = \mathbb{Q}^c$ → unc'ble
4) $A\cup B = \mathbb{Q}\cup\mathbb{Q}^c = \mathbb{R}$ → unc'ble

Result:- ~~Card~~ cartesian Product of two c'ble set is c'ble.

Eg:- $\mathbb{N}\times\mathbb{N}$, $\mathbb{Q}\times\mathbb{Q}$, $\mathbb{Q}\times\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$ all are c'ble

Ex:- Given $[0,1]$ is unc'ble. Prove that $\mathbb{R}$ is unc'ble.

Pf $[0,1]$ is subset of $\mathbb{R}$ and $[0,1]$ is unc'ble.
'A set has unc'ble subset, then set is unc'ble'
⇒ $\mathbb{R}$ is unc'ble.

Q4 Prove that $\mathbb{Q}^c$ is unc'ble.
Let $\mathbb{Q}^c$ is c'ble. and we know that $\mathbb{Q}$ is c'ble so by (2) c'ble union of c'ble set is c'ble.
⇒ $\mathbb{Q}\cup\mathbb{Q}^c = \mathbb{R}$ is c'ble.
which is contradication
So our supposition is wrong.
⇒ $\mathbb{Q}^c$ is unc'ble.

Let A be any set. If set A has subset the set of all irrational Number. then set A is unc'ble i.e $\mathbb{Q}^c \subseteq A$

Note:- set of all irrational Number is Necessary.

: If we take only some set of Irrational Number i.e

$B = \{\sqrt{p} ; p \text{ is prime}\}$

then ; clearly B is c'ble.

So B being subset of $\mathbb{Q}^c$ is c'ble.

∵ $A = \{p ; p \text{ is prime}\} \subset \mathbb{N}$

$f : A \longrightarrow B = \{\sqrt{p} ; p \text{ is prime}\}$

$f(p) = \sqrt{p}$.

Clearly f is one-one & onto.

$\Rightarrow A \sim B$.

$\Rightarrow$ B is c'ble.

Hence all word is must.

Result:- set of all Polynomial of degree $\leq n$ whose coefficient are integers, then set is c'ble.

$S = \{P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n ; a_i \in \mathbb{Z}\}$ c'ble.

Result:- Set of all Polynomial of degree 'n' whose coefficients are Rationals, then set is c'ble.

$S = \{P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n ; a_i \in \mathbb{Q}\}$ is c'ble.

Result:- set of all Polynomial of degree 'n' whose coefficients are Real, then set is unc'ble.

$S = \{P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n ; a_i \in \mathbb{R}\}$ is unc'ble.

Result:- set of all circles in coordinate plane $(\mathbb{R}^2)$ whose centre are Rational and Raddi are Rational, then Set is c'ble.

Result:- Set of all concentric circles with Rational Raddi is c'ble

Result set of all unit circles whose centres are Rational is c'ble.

$|S| = \mathbb{Q}\times\mathbb{Q}$ c'ble. ie $S = \{(x-a)^2+(y-b)^2=1 \; ; \; a, b \in \mathbb{Q}\}$ c'ble.

Result. Set of all disjoint open interval is c'ble.

Result set of all coordinates in $\mathbb{R}^2$ and coordinates are Rational is c'ble.

or, $S = \{(a,b) \in \mathbb{R}^2 : a, b \in \mathbb{Q}\} = \mathbb{Q}\times\mathbb{Q}$ is c'ble.

or, set of all Non-overlapping open intervals is c'ble.

Result Set of all coordinates in $\mathbb{R}^2$ and coordinates are integer is a c'ble.

or; $S = \{(a,b) \in \mathbb{R}^2 \; ; \; a, b \in \mathbb{Z}\} = \mathbb{Z}\times\mathbb{Z}$ is c'ble.

Result set of all coordinates in $\mathbb{R}^2$ and coordinates are Real is unc'ble.

or $S = \{(a,b) \in \mathbb{R}^2 \; ; \; a, b \in \mathbb{R}\} = \mathbb{R}\times\mathbb{R}$ is unc'ble.

Result Set of all straight lines passing through the origin is ~~un~~ unc'ble

$y = mx, \; m \in \mathbb{R}$ is unc'ble.

Result $S = \{\alpha + \pi \; ; \; \alpha \in \mathbb{Q}\}$ is c'ble.

Result $S = \{\alpha\pi \; ; \; \alpha \in \mathbb{Q}\}$ is c'ble.

Result The set of all Algebraic Number is c'ble.

Result The set of all Transcendental Number in $(0,1)$ is unc'ble.

Algebraic Number :- Any Number $\alpha \in \mathbb{R}$ is called Algebraic Number if $\exists$ a Poly $p(x) \in \mathbb{Q}[x]$ such that $p(\alpha) = 0$

Eg:- $\sqrt{2}$ is an Algebraic Number as $\sqrt{2}$ is Root of $x^2-2=0$

Note:- Any Number of the form $\sqrt[p]{p/q}$ is Algebraic

E.g:- $\sqrt{3}$, $\sqrt[3]{5/3}$, $\sqrt[5]{1/4}$ all are algebraic Number.

Trancedental Number :- A Real Number which is Not Algebraic in $\mathbb{R}$ is called transcendental Number.

Trancedental Number $\Rightarrow$ Irrational Number.
Rational Number $\Rightarrow$ Algebraic Number.

Example for Not Holding converse are:-

$a = \sqrt{2} \in \mathbb{Q}^c$
and $\sqrt{2}$ is Root of $x^2-2=0$
$\Rightarrow$ $\sqrt{2}$ is Not ~~tranceenterl~~ ~~N~~ transcedental Number
So Irrational $\not\Rightarrow$ Trancedental Number.

Similarly, $\sqrt{2}$ is Root of $x^2-2=0$
$\Rightarrow$ $\sqrt{2}$ is Algebraic Number.
But $\sqrt{2}$ is Not Rational.
So Algebraic $\not\Rightarrow$ Rational Number.

Discuss the Behaviour of A and B if

a) $f: A \to B$ is onto & A is c'ble.

Ans:- $\because$ f is onto $\Rightarrow$ $|A| \geq |B|$
$\Rightarrow$ $|B| = \aleph_0$ or finite i.e B is c'ble.

b) f : A → B is onto and B is unc'ble.

∵ f is onto |A| ≥ |B| & B is unc'ble

So A must be unc'ble.

c) f : A → B is one-one and A is c'ble then |B| ≥ |A| ⇒ B can be countable or unc'ble.

d) f : A → B is one-one & B is c'ble.
⇒ |B| ≥ |A| ⇒ A can be finite or countable.

e) f : A → B is one-one and A is unc'ble
⇒ |B| ≥ |A|,
⇒ B must be unc'ble.

Arithmetic of Transfinite Number :-

Defn:- finite cardinal Numbers :- A cardinal Number corresponding to a finite set is called finite cardinal Number.

Defn:- Transfinite cardinal Numbers :- A cardinal No. corresponding to an infinite set is called transfinite cardinal number.

1) Extension of continuum Hypothesis :-
There is No transfinite Number between $\aleph_0$ and $2^{\aleph_0}$ Here $\aleph_0 = |\mathbb{N}| =$ card($\mathbb{N}$) and $2^{\aleph_0}$ $= c =$ card ($P(\mathbb{N})$)
i.e There is No cardinal Number b/w λ and 2^{λ}, where λ is any transfinite Number and 2^{λ}

is cardinality of Power set of that set whose cardinality is λ.

2> if $|A| = \lambda$ & $|B| = \mu$ then

a> $|A \times B| = \mu \cdot \lambda = \lambda \; (\lambda > \mu)$ $[A \times B \sim A$ if $\lambda > \mu]$
b> $|A \times B| = \lambda \cdot \mu = \mu \; (\mu > \lambda)$ $[A \times B \sim B$ if $\mu > \lambda]$
c> $|A \cup B| = \lambda + \mu = \lambda$ (if $\lambda > \mu$)
d $|A \cup B| = \lambda + \mu = \mu$ (if $\mu > \lambda$)

3> Cancellation law does Not exist

if $x_0 + x_0 = x_0$
if cancellation law hold, then
$x_0 = 0$ which is Not possible.

4> Exponent Rule is valid.

$$(2^{\lambda})^{\mu} = (2^{\mu})^{\lambda} = 2^{\lambda \mu}$$

*> The set of all intervals with Rational end points is c'ble.

*> Set of isolated points is c'ble.
*> Derived set of unc'ble set is unc'ble.

Note → Derived of countable may be anything.

Eg:- $\mathbb{Q}' = \mathbb{R}$
$\mathbb{N}' = \phi$
$S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \Rightarrow S' = \{0\}$

Note:- If set has finite Number of limit points then set is c'ble.

1) set of transcedental Number over Q is unc'ble.

2) let $A_1, A_2, \ldots, A_n, \ldots$ are countably infinite set. Then, $A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots$ is c'ble infinite set

3) Let $A_1, A_2, \ldots, A_n$ are countable infinite sets, then $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ is c'bley infinite.

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Algebra of cardinal Number.

1) $a + \aleph_0 = \aleph_0$; a is finite cardinal No.

2) $a \cdot \aleph_0 = \aleph_0$

3) $\aleph_0 + c = c$

4) $a \cdot c = c$

5) $\aleph_0 \cdot c = c$

6) $a + c = c$

7) $\aleph_0^{\aleph_0} = c$

8) $2^{\aleph_0} = c$

9) $\cancel{\aleph_0^{\aleph_0}} = c^{c} = c$

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Notes:- Set of all sequences of set $\{0,1\}$ is unc'ble

or $f : A \to B$

$A = \{ f : f : N \to \{0,1\} \}$

$|A| = |B|^{|A|}$

$|A| = 2^{\aleph_0} = c$ unc'ble.

2) Set of all sequences of set A and $|A| > 1$ is always unc'ble.

our $S = \{ f : \mathbb{N} \to A \}$.

$|S| = |A|^{\aleph_0} = \mathfrak{c}$ unc'ble.

3> set of all sequences on $A = \{a\}$ is c'ble.

$S = \{ f : f : \mathbb{N} \to A \}$.

$|S| = |A|^{\aleph_0} = 1^{\aleph_0} = 1$ c'ble

4> Set of all functions from $\mathbb{N}$ to set A, $|A| \geq$
, is always unc'ble.

5> set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is unc'ble.

$S = \{ f : f : \mathbb{N} \longrightarrow \mathbb{N} \}$

$|S| = \aleph_0^{\aleph_0} = \mathfrak{c}$ unc'ble.

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6> Set of all functions from.

$\mathbb{N}$ to $\mathbb{Z}$ $\to$ $\mathfrak{c}$ $\to$ unc'ble

$\mathbb{R}$ to $\mathbb{R}$ $\to$ $\mathfrak{c}$ $\to$ unc'ble.

$\mathbb{N}$ to $\mathbb{R}$ $\to$ $\mathfrak{c}$ $\to$ unc'ble

$\mathbb{Z}$ to $\mathbb{Z}$ $\to$ $\mathfrak{c}$ $\to$ unc'ble.

$$\left\{ \begin{array}{l} \aleph_0^{\aleph_0} = \mathfrak{c} \\ \mathfrak{c}^{\mathfrak{c}} = \mathfrak{c} \\ \mathfrak{c}^{\aleph_0} = \mathfrak{c} \\ \aleph_0^{\aleph_0} = \mathfrak{c}. \end{array} \right.$$

7> Set of all sequences on $\mathbb{Z}$ which is eventually zero is c'ble.

8> Set of all sequences with entries 0 and 1 is unc'ble.

Eventually zero :- A function f is called eventually zero if $\exists$ a +ve integer m such that $f(n)=0$ $\forall\ n\geq m$.

Eventually one :- $f(n)=1$ $\forall\ n\geq m$

Eventually two :- $f(n)=2$ $\forall\ n\geq m$

9> Set of all characteristic functions on $X=\mathbb{R}$ is always unc'ble.

Characteristic functions:-

$$\chi_A(x)=\begin{cases}0 & \text{if } x\notin A\\ 1 & \text{if } x\in A\end{cases}$$

where $A\subset X=\mathbb{R}$.

11> $S=\{f : f: A\to Z\}$ where $A=\{0,1\}$

$$|S|=\aleph_0^{|A|}=\aleph_0^{2}=\aleph_0\cdot\aleph_0=\aleph_0\to \text{c'ble}$$

12> Cantor's set is Non-denumerable (unc'ble).

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POINT SET OF TOPOLOGY.

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Neighbourhood :- Let S be subset of Real Number, is a Nbd of point x if $\exists$ open interval I such that $x \in I \subseteq S$

Note :- S is Nbd of 'a' $\in \mathbb{R}$ if $\exists$ $\delta > 0$ s.t $(a-\delta, a+\delta) \subseteq S$.

Note :- Let $a \in \mathbb{R}$ then if anyone ask the Nbd of 'a' then we can construct always Nbd of a.

Let $a \in \mathbb{R}$, for any $\delta > 0$
$S = (a-\delta, a+\delta)$ open interval
then this δ will be Nbd of 'a' $\forall$ $\delta > 0$

In Particular :- $\left(a - \frac{1}{n}, a + \frac{1}{n}\right)$ is Nbd of 'a'.

Properties :-

1) Since a set S which is Nbd of some point then S will contain an interval, and interval is uncble Hence, S is uncble.

Note :- Finite set and countably Infinite set can't be Nbd of any Real Number.

Eg:- $\{1,2,3,4\}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ can't be Nbd of any Real Number.

Notes:- Any set which doesn't contain an interval then set can't be Nbd of any Real Number i.e. uncountability only Necessary condition Not sufficient for a set to be Nbd of some point.

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Eg:-1) $\mathbb{Q}^c$ which is uncble but Not Nbd of any Real Number because It does N't contain any interval.

2) Every Interval is Nbd of all of its point only Except the sup and inf of interval. Hence open interval is Nbd of all of its point.

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3) The Null set ϕ is Nbd of each of its point in a sense that there is No point in ϕ of which it is Not a Nbd

4) Superset of a Nbd of point 'x' is also a Nbd of 'x'.

5) Union (finite or Arbitary) of Nbd of a point x is again a Nbd of x.

6) if M and N are two ~~Nbd~~ Nbds of a point x.

then $(M \cap N)$ is also a Nbd of α
You can generalize uto n terms.

Note:- Set of Real Number $\mathbb{R}$ is Nbd of all of its point.

Deleted Neighbourhood :- Let S be a subset of Real Number is called Deleted Neighbourhood of x if $\exists$ an open interval sot

$$x \notin I \subset S$$

or

$$(x-\epsilon, x+\epsilon) / \{x\} \subset S$$

Open set :- Let S be a subset of $\mathbb{R}$ is called open if it is Nbd. of each of its point.

or

Every point of set S is interior point.

Which of the following is open or Not.

Eg 1) $S_1 = (a,b] \Rightarrow b \in (b-\epsilon, b+\epsilon) \notin [a,b]$, Not Nbd of b ∴ Not open

2) $S_2 = [a,\infty) \Rightarrow a \in (a-\epsilon, a+\epsilon) \notin [a,\infty)$, Not a Nbd of '$a$'. ∴ Not open

3) $S_3 = \mathbb{R}$ is Nbd of all of its points. Hence open.

4) $S_4 = \left\{\frac{1}{n} ; n \in \mathbb{N}\right\}$ Not Nbd of any of its point.
∴ Not open

5) $S_5 = \left\{\frac{1}{n} ; n \in Z-\{0\}\right\}$ Not Nbd of any of its point.
∴ Not open

6> $S_6 = \{x \in \mathbb{R} ; x \in \mathbb{Q} \cap [0,1]\}$
Not Nbd of any it's point. Since $\mathbb{Q} \cap [0,1] =$ Rational No. b/w $[0,1]$ Not Nbd of any of its points. ∴ Not open

7> $S_8 = \left\{ 1 + \frac{1}{n} ; n \in \mathbb{N} \right\}$

Not Nbd of any of its point

8> Arbitrary union of open set is open
9> Finite Intersection of open set is open

10> Arbitrary Intersection of open set is Not open.

$$A = \left(\frac{-2}{n}, \frac{2}{n} \right)$$

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$

$\{0\}$ is closed set.

Limit point / cluster point / accumulation :-

A Real Number x (belong to set S or Not) is called a limit point of set S if every Nbd of point x contain atleast one point of set S other than x.

or

Every Nbd of point x contain infinite point of set S

or

Every Nbd of point x contain more than one

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point of set S.

or.

Every deleted Nbd of point x contain atleast one point of set S.

i.e $S \cap (x-\epsilon, x+\epsilon) - \{x\} \neq \phi$ $\epsilon \in$

or $W_x \cap S - \{x\} \neq \phi \quad \forall \; W_x$.

W_x is the Nbd of x.

Eg :- $S_1 = (0,2)$, $S_2 = [0,2]$

0 and 2 and all points b/w 0 & 2 are limit points of set S_1 and S_2. In S_2 0 & 2 belong to S_2 but 0 & 2 does not belong to S_1

Derived set :- A set is called derived set if it contain all of its limit point. It is denoted by S' or $D(S)$.

Notes:- finite set has No limit point as Every Nbd of limit point will contain infinite member of set but set has only finite points.

Note:- Infinite set Need Not have limit point.
i.e Infiniteness only Necessary condition Not sufficient for a set to have a limit point.
Eg:- $\mathbb{N}$, $\mathbb{Z}$, $S = \{(-1)^n n \; ; \; n \in \mathbb{N}\}$.

Note:- Limit point may or may Not be belongs to set.

Examples

1) $S_1 = \mathbb{N}$, $S' = \phi$

2) $S_2 = \mathbb{Z}$; $S' = \phi$

3) $S_3 = \mathbb{Q}$; $S' = \mathbb{R}$

Every limit point of the set $\mathbb{Q}$ is a limit point, for b/w any two Rationals there exist an infinity of Rationals. further Every irrational Number is also a limit point of $\mathbb{Q}$. for b/w any two irrationals there are infinitely many Rationals. Thus every Real Number is a limit point of $\mathbb{Q}$. So that $\mathbb{Q}' = \mathbb{R}$.

4) $S_4 = \{\frac{1}{n} ; n \in \mathbb{N}\}$

$S_4' = \{0\}$.

$(0-\epsilon, 0+\epsilon) - \{0\} \cap S_4 \neq \phi$

or

$\lim_{n\to\infty} \frac{1}{n} = 0$ = limit ⇒ limit point

$S' = \{0\}$.

limit ⇒ limit point

But limit point ⇏ limit

5) $S_5 = \{\frac{1}{n} ; n \in \mathbb{Z} - \{0\}\}$

$S_5' = \{0\}$.

The set S_5 has only one limit point, zero, which is Not a member of the set S_5.

6) $S_6 = \left\{ \frac{1}{n} ; n \in Q^* \right\}$

$S_6' = \mathbb{R}$.

7) $S_7 = \left\{ -1 + \frac{1}{n} ; n \in Z^* \right\}$; $S_7' = \{-1\}$

8) $S_8 = \{ x \in Q ; x \in Q \cap [0,1] \}$

$S_8' = \{ x : x \in [0,1] \}$

9) $S_9 = \{ x \in (-2,1) ; x \in Q^c \}$

$S_9' = \{ x ; x \in [-2,1] \}$

10) $S_{10} = \{ x ; x < \infty \} = (-\infty, x)$

$S_{10}' = (-\infty, x]$

11) $S_{11} = \left\{ \pm \frac{1}{n} ; n \in Q^c \right\}$

$S_{11}' = \mathbb{R}$.

12) $S_{12} = \left\{ 1 + \frac{1}{n} ; n \in \mathbb{N} \right\}$

$S_{12}' = \{1\}$

13) $S_{13} = \left\{ i + \frac{1}{n} ; n \in \mathbb{N} ;\ i \text{ is a fix. No.} \right\}$

$S_{13}' = \{i\}$

14) $S_{14} = \left\{ \frac{1+(-1)^n}{n} ; n \in \mathbb{N} \right\}$

$$S_{14} = \begin{cases} 0 & ; n \text{ is even} \\ \frac{2}{n} & ; n \text{ is odd} \end{cases}$$

$S' = \{0\}$.

15) Derived set of set of Algebraic Numbers is $\mathbb{R}$.

16) Derived set of set of all transcendental Numbers is $\mathbb{R}$.

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Special Result

Let $S = \{x + \frac{1}{n} ;\ x \in A,\ n \in \mathbb{N}$ where $A \subseteq \mathbb{R}\}$

then Derived set $S' = A \cup A'$.

Application :-

1) $S = \{m + \frac{1}{n} ;\ m, n \in \mathbb{N}\}$

Here $A = \mathbb{N}$ so $S' = \mathbb{N} \cup \mathbb{N}' = \mathbb{N}$

2) $S = \{x + \frac{1}{n},\ n \in \mathbb{N},\ x \in \mathbb{Q}\}$

$S' = A \cup A' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$

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3) $S = \{\frac{1}{m} + \frac{1}{n} ;\ m, n \in \mathbb{N}\}$

Here $A = \{\frac{1}{m},\ m \in \mathbb{N}\}$ or $\{\frac{1}{n},\ n \in \mathbb{N}\}$

So $S' = A \cup A'$

$= \{\frac{1}{m},\ m \in \mathbb{N}\} \cup \{0\}$

or $\{\frac{1}{n},\ n \in \mathbb{N}\} \cup \{0\}$.

Note:- If & x and n both are same and belong to $\mathbb{N}$ then Result is Not applicable.

Eg: $S = \{n + \frac{1}{n} , n \in \mathbb{N}\}$

then $S' = \phi$.

Some important Examples:-

1) $S = \{m + n\sqrt{2} ; m, n \in \mathbb{N}\}$ then
$S' = \phi$

2) $S = \{m + n\sqrt{2} ; m, n \in \mathbb{Z}\}$ then
$S' =$ contact S.K.M Academy "8199903932"

3) $S = \{m + n\sqrt{2} ; m, n \in \mathbb{Q}\}$
then $S' = \mathbb{R}$

4) $S = \{\sin\frac{1}{n} ; n \in \mathbb{N}\}$

then $S' = \{0\}$

5) $S = \{x + \frac{1}{n} ; x = \frac{p}{q} \, (p, q \in \mathbb{N}) \; \& \; n \in \mathbb{N}\}$ then

$S' = \{\frac{p}{q} , p, q \in \mathbb{N}\} \cup \{\frac{p}{q} ; p, q \in \mathbb{N}\}'$

$= [0, \infty)$

Result:- The Derived Set of a Bounded Set is Bounded.

Note:- If S is bounded So its ~~so~~ closure is bdd.

Note:- If sup or inf of S is Not member of S then it is limit point and in view of the

thm " Derived set of Bounded set is bdd"
it is the greatest or least member of S'.

Note :- If Sup M and Inf. m belongs to set S, then it is Not Necessarily a limit point of S.
i'e M and m Not always be Supermum and Infimum of S' but always for $\bar{S} = S \cup S'$

Eg :- $S = \left\{ (-1)^n \left[1 + \frac{1}{n} \right], n \in \mathbb{N} \right\}$

$S = \left\{ ---, -1\frac{1}{5}, -1\frac{1}{3}, -2, 1\frac{1}{2}, 1\frac{1}{4}, 1\frac{1}{6}, --- \right\}$

Here $m = -2$, $M = \frac{3}{2} \left(1\frac{1}{2} \right)$

1 and -1 are two limit points of S.

i'e $S' = \{1, -1\}$

$\text{Inf } S' = -1 \neq m$ and $\text{Sup } S' = 1 \neq M$.

but $\text{Inf } \bar{S} = -2 = m$, $\text{sup } \bar{S} = 1\frac{1}{2} = M$.

Note :- The derived set S' of ~~the~~ a bdd infinite set $S (\subseteq \mathbb{R})$ has smallest and the largest member.

# Closed set :- A set $S \subseteq \mathbb{R}$ is called closed set if its contains all of its limit point.

or $\bar{S} = S$

or. Derived Set is subset of itself
$\bar{S} = S \cup S'$

1) $\mathbb{N}$, $\bar{\mathbb{N}} = \mathbb{N} \cup \mathbb{N}' = \mathbb{N} \cup \phi = \mathbb{N} \Rightarrow \mathbb{N}$ is closed.
2) $\mathbb{Z}$, $\bar{\mathbb{Z}} = \mathbb{Z} \cup \mathbb{Z}' = \mathbb{Z} \Rightarrow \mathbb{Z}$ is closed.
3) $\mathbb{R}$, $\bar{\mathbb{R}} = \mathbb{R} \cup \mathbb{R}' = \mathbb{R} \Rightarrow \mathbb{R}$ is closed
4) $(0,1)$, ~~$\bar{\phi}$~~ $\overline{(0,1)} = (0,1) \cup [0,1] = [0,1] \Rightarrow$ Not closed.
5) $\mathbb{Q}$, $\bar{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R} \Rightarrow$ Not closed
6) $S = \{\frac{1}{n} ; n \in \mathbb{N}\}$, $\bar{S} = \{\frac{1}{n} ; n \in \mathbb{N}\} \cup \{0\} \Rightarrow$ Not closed

7) $S = \{x \in (0,1) ; x \in \mathbb{Q}\}$, $\bar{S} = [0,1] \Rightarrow$ Not closed.
8) $[0,1] \cup [2,3]$ {Not an interval} is closed set.

Note:- Every closed interval is closed set.
Note:- Every finite set is closed set.
Note:- A set whose derived set is ϕ then set is closed.

Adherent point:- Let $S \subseteq \mathbb{R}$, A Real Number 'x' is called the adherent point of set S if every Nbd of 'x' contains point of set S

i.e $W_x \cap A \neq \phi \quad \forall \; W_x$.
or $(x-\delta, x+\delta) \cap A \neq \phi$

i.e

Every Nbd of 'x' contains atleast one member of A.

Note:- It is clear from the defination if $a \in A$ then $\forall \; \delta > 0$, $(a-\delta, a+\delta) \cap A \neq \phi$ then we can say. Every member of set is adherent point. But Not converserly.
(adherent point Need Not be member of set)

Eg:- 1) $S = \{1,2,3\}$.
Here $(1-\delta, 1+\delta) \cap S \neq \phi \ \forall \ \delta > 0$. Hence 1 is adherent and $1 \in S$ similary 2 & 3 are adherent Because all are belongs to S.

2) $(0,1)$, here 0 & 1 are adherent point but $0, 1 \notin S$.

Note:- The set of all adherent points of S consists of S and its derived set S'.

Closure of a set:- The set of all adherent points called closure of set S, denoted by $\bar{S}$
i.e $\bar{S} = S \cup S'$

Isolated point:- Let $S \subseteq \mathbb{R}$ and $a \in S$ we say a is an isolated point of S if $\exists$ a Nbd of a which does not contain any element of S other than 'a'.

i.e A mememberwhich is Not a limit point is called isolated point.

i.e. let $a \in S$ and if $\exists \ \delta > 0$ s.t

$$(a-\delta, a+\delta) \cap S - \{a\} = \phi$$

$$(a-\delta, a+\delta) \cap S = \{a\}.$$

Note:- Finite set has no limit point hence Every member of finite set is isolated point.

Note:- Every member of an infinite set may be isolated.

Eg:- $\mathbb{N}$, Z', $S = \{\frac{1}{n} ; n \in \mathbb{N}\}$

$\mathbb{N}$, Z has No limit point Hence all Member are isolated and in S 0 is limit point (only) but $0 \notin S$, Hence every member of S is isolated point

Note:- If S be any set and S' its Derived set then $S - S'$ give set all isolated point of S

Eg:- $Q - Q' = \phi$

i.e Q has No isolated point.

Note:- If set has No Isolated point then set is infinite or empty set

Adherent point

Isolated point

Limit point

Interior point:- Let $S \subseteq \mathbb{R}$ and 'a' $\in \mathbb{R}$ we say 'a' is an interior point of S if $\exists$ $\delta > 0$ s.t $(a-\delta, a+\delta) \subset S$ i.e S is Nbd of a.

Results:- 1) if a is an interior point of S then $a \in S$ Not conversely.

Eg:- $S = [0,1]$ here $0, 1 \in [0,1]$ but are Not interior points.

2) If a is a interior point of S then S is Nbd of 'a' and conversely.

3) set of all interior points of set S is denoted by S° or int (S)

E.g $\mathbb{R}^\circ = \mathbb{R}$

$\mathbb{N}^\circ = \phi$

$Z^\circ = \phi$

$(a,b)^\circ = (a,b)$

$[a,b]^\circ = (a,b)$.

4) Int (S) or S° is always subset of S.

i.e $S^\circ \subset S$.

Note:- A set is said to be open if $S^\circ = S$

Ex – $\mathbb{R}^\circ = \mathbb{R}$, $\phi^\circ = \phi$

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Results

1) A set may be Neither open nor closed.

Eg Q, $[0,1)$, $S = \{\frac{1}{n}, n \in \mathbb{N}\}$

2) IR and ϕ are only two sets which are open and closed both.

3) Uncountability or an infinite is Neccessary condition for open set Not sufficient.

open set $\Rightarrow$ unc'ble set

unc'ble set $\not\Rightarrow$ open set

Eq :- Q^c is unc'ble but Not open.

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4) if we Remove finite No. of elements from open set it Remains open but Not closed

5) Derived set of any interval except ϕ is unc'ble.

Exterior point :- A Real Number x is called an Exterior point of set S if S^c is Nbd of x i.e $\exists\ \epsilon>0$ s.t $x \in (x-\epsilon, x+\epsilon) \subseteq S^c$

Note. Exterior of set S is $\operatorname{int}(S^c)$

Boundary point :- Let $A \subseteq \mathbb{R}$. A point x is called Boundary point of set A if x is Neither interior point Nor Exterior point of A.

Eg: $A = \{1\}$, find Boundary point of A.

$A = \{1\}$

$A^\circ = \phi$

$(A^c) = (-\infty, 1) \cup (1, \infty)$

$(A^c)^\circ = (-\infty, 1) \cup (1, \infty)$

$A\ \{1\} \notin A^\circ \cup (A^c)^\circ$

$\Rightarrow \{1\}$ is Boundary point.

i.e $A^{bd} = \{1\}$.

Eg:- $A = \{1, 2\}$, find Boundary point.

$A^\circ = \phi$

$A^c = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$

$(A^c)^\circ = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$.

$\{1, 2\} \notin A^\circ \cup (A^c)^\circ$

$\Rightarrow A^{bd} = \{1, 2\}$.

Eg: $A = \mathbb{Q}$, find Boundary point.

$\mathbb{Q}^\circ = \phi$

$(\mathbb{Q}^c)^\circ = \phi$

$\mathbb{R} \not\subset A^\circ \cup A^{ext}$

$\Rightarrow A^{bd} = \mathbb{R}$

$\Rightarrow$ Boundary of $\mathbb{Q}$ is $\mathbb{R}$.

Note:- Interior of finite set is empty

Eg:- $A = [1,2) \cup \{3\}$ find Boundary

$A^\circ = (1,2)$

$A^c = (-\infty,1) \cup [2,3) \cup (3,\infty)$

$(A^c)^\circ = (-\infty,1) \cup (2,3) \cup (3,\infty)$

$A^{Bd} = \{1,2,3\}$.

Discrete set:- A set S is said to be Discrete set if its Derived set is empty.

Eg:- Set of $\mathbb{N}$ No. and $\mathbb{Z}$ are Discrete

$N' = \phi$

$Z' = \phi$

Some - Important Results on point set Topology:-

1) The Union of arbitary family of open sets is open set.

2) finite intersection of open set is open [Not true for arbitrary intersection].

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Eg. $I_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$

$\bigcap_{n \in \mathbb{N}} I_n = \{0\}$. which is Not an open set.

3> for any set A, A° is open.
4> A° is largest open set contained in A
5> if $A \subset B$ then $A^\circ \subset B^\circ$
6> A is open iff $A^\circ = A$ and $(A^\circ)^\circ = A$.
7> finite Union of closed set is closed
8> Arbitary Intersection of closed set is closed.
9> Arbitary union of closed set is Need Not be open.

Ex $I_n = \left[\frac{1}{n}, 1\right]$ $\forall$ $n \in \mathbb{N}$

$\bigcup_{n=1}^{\infty} I_n = (0, 1]$, which is Not closed set.

10> Bolzano-Weirstrass Theorem for set :-
Every Infinite and bdd subset of $\mathbb{R}$ has a limit point.

11> if $A, B \subseteq \mathbb{R}$ then.
I> $A \subseteq B \Rightarrow A' \subseteq B'$
II> $(A \cup B)' = A' \cup B'$
III> $(A \cap B)' \subseteq A' \cap B'$

12> ⇒ if $A, B \subseteq \mathbb{R}$ then.
1> $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$
2> $(A \cap B)^\circ = A^\circ \cap B^\circ$
3> $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ = (A \cup B)^\circ$

Eg :- $A = (0, 1]$ and $B = (1, 2)$.

$A \cup B = (0,2)$

$(A \cup B)^\circ = (0,2)$

$A^\circ = (0,1)$, $B^\circ = (1,2)$

$A^\circ \cup B^\circ = (0,1) \cup (1,2)$

$A^\circ \cup B^\circ \subset (A \cup B)^\circ$

Eg: $A \cap B = \phi$

$A = (0,1]$ and $B = (1,2)$

$(A \cap B)' = \phi$

and $A' = [0,1]$, $B' = [1,2]$.

$A' \cap B' = \{1\}$.

$(A \cap B)' \subset A' \cap B'$.

13) A is closed iff $A = \overline{A}$

14) For any set A, A' is closed set.

15) For any set A, $\overline{A}$ is closed set.

16) if $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

17) For any set A, $\overline{A}$ is smallest closed set containing A

i.e $A \subset \overline{A}$

18) Every Bdd Non-empty closed set contains it's Supremum and Infimum.

19) a) $\overline{(\overline{A})} = \overline{A}$

b) $(\overline{A \cup B}) = \overline{A} \cup \overline{B}$

c) $(\overline{A \cap B}) = \overline{A} \cap \overline{B}$

20) if $A \subset R$, then $\overline{A}$ is the intersection of all closed set containing A.

21) Every open set is union of open intervals.

22) The interior of a set S is largest open subset of S

23> Cartesian Product of two open sets is open.

24> Cartesian Product of finitely many closed set is closed.

26> Sum of two closed set Need not be closed

27> Sum of two open sets in $\mathbb{R}^n$ is open have

Sum

$S_1+S_2 = \{v_1+v_2 ; v_1 \in S_1, v_2 \in S_2\}$.

28> A is open iff ~~AU~~ $A \cap A^{bd} = \phi$

29> A is open $\Rightarrow$ A^c is closed.

Which of the following statement is true for set $A \subseteq \mathbb{R}$

1> Limit point $\Rightarrow$ Adherent point $\boxed{T}$

2> Adherent point $\Rightarrow$ Limit point $\boxed{F}$

3> Boundary point $\Rightarrow$ Isolated point $\boxed{F}$

4> Interior point $\Rightarrow$ Limit point $\boxed{T}$

5> Limit point $\Rightarrow$ Boundary point $\boxed{F}$

6> Isolated point $\Rightarrow$ Boundary point $\boxed{T}$

7> Interior point $\Rightarrow$ Adherent point $\boxed{T}$

8> Boundary points of A and A^c are same $\boxed{T}$

9> if x is Boundary point of A and $x \notin A$ $\boxed{T}$

$\Rightarrow$ x is limit point

10> if x is limit point of A & $x \notin A$ $\Rightarrow$ x is bd. point $\boxed{T}$

5> $S = [2,5]$

Here 4 is limit point of S but Not Boundary point.

$S^{bd} = \{2,5\}$.

7> $\mathbb{R}^\circ = \mathbb{R}$ and $\mathbb{R}$ has Adherent point.

But Not conversely of this Result.

$\mathbb{Q}$ has Adherent point But $\mathbb{Q}^\circ = \phi$.

9> $\sqrt{3}$ is Bd point of $\mathbb{Q}$ and $\sqrt{3} \notin \mathbb{Q}$ $\Rightarrow$ $\sqrt{3}$ is limit point of $\mathbb{Q}$.

A point $x \in \mathbb{R}$ is called limit point of set A if $\exists$ a seq^n $\langle x_n \rangle$ of A other than x and $x_n \longrightarrow x$

Closed set :- A set A is said to be closed set if for all seq^n $\langle x_n \rangle$ of A and $\lim x_n \in A$.

Open set :- A set A is said to be open set if $\nexists$ seq^n x_n of A^c and $x_n \rightarrow x$ and $x \in A$

Dense set :- Let A and B are two subsets of $\mathbb{R}$ s.t $A \subset B$. Then the set A is said to be Dense (or everywhere dense) in B if $B \subset \bar{A}$.

Dense in ~~int~~ itself :- A set A is said to be Dense in itself if every point of A is limit point of A i.e $A \subseteq A'$.
Or A set A is said to be Dense in itself if A is contained in it's Derived set or $A \subseteq A'$.

Dense in $\mathbb{R}$ / Everywhere Dense :- A set A is said to be Dense in $\mathbb{R}$ iff $\bar{A} = \mathbb{R}$.

Perfect set :- A set is said to be Perfect set if $A \subseteq A'$ and $A' \subset A$ or dense itself i.e $A = A'$.
Eg :- $\mathbb{R}' = \mathbb{R}$, $\phi' = \phi$, $[a,b]' = [a,b]$.

Nowhere Dense set :- A set A is said to be Nowhere Dense set if its closure has Empty interior.

i.e $(\bar{A})^\circ = \phi$

(or)

A set A Nowhere dense if the complement of the closure of A is dense in $\mathbb{R}$

Eg:- $A = \{1\}$.

$A^\circ = \phi$

$\bar{A} = A = \{1\}$.

$(\bar{A})^\circ = \phi$

$\Rightarrow$ A is Nowhere dense set.

$A = \{1\}$.

$A' = \phi$

$A \not\subset A' \Rightarrow$ Not dense in itself.

2) $A = \mathbb{N}$

$A^\circ = \phi$

$A' = \phi$

$\bar{A} = \mathbb{N} \neq \mathbb{R}$ Not dence in $\mathbb{R}$

$(\bar{A})^\circ = (\mathbb{N})^\circ = \phi \Rightarrow$ Nowhere dense set.

$\mathbb{N} \nsubseteq \phi \Rightarrow$ Not Perfect, Not dense in itself.

3) $A = [a,b)$

$\bar{A} = [a,b]$

$A' = [a,b]$

$A^\circ = (a,b)$

$[a,b) \subseteq [a,b]$ dense in itself

$[a,b] \nsubseteq [a,b)$ Not Perfect

$(\bar{A})^\circ = ([a,b])^\circ = (a,b)$ Not Nowhere dense set-

4> $A = \mathbb{R}$

$A^\circ = \mathbb{R}$

$\bar{A} = \mathbb{R}$

$A' = \mathbb{R}$

$A \subset A'$ Dense itself.

$A \subset A'$ and $A' \subset A \Rightarrow$ Perfect set.

$(\bar{A})^\circ = (\mathbb{R})^\circ = \mathbb{R}$ Not Nowhere dense set.

Any open interval is open set

Eg = ϕ

let A and B are subset of $\mathbb{R}$ and A is closed and B is open then.

1> $A - B$ is closed

2> $B - A$ is open

3> $A \cup B$ Not open Not closed

4> $A \cap B$ Nothing to say.

$A = \{1\}$

$B = (2,3)$

$A \cup B = (2,3) \cup \{1\}$.

$A = \{1, 1.5, 2\}$

$B = (1,2)$

$A \cap B = \{1.5\}$ closed

$A = [1,2]$

$B = (1,2)$

$A \cap B = (1,2)$ is open

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so $A \cap B$ Nothing to say.

Find Derived set:

1) $S_1 = \left\{ -1 + \frac{1}{n} ; n \in \mathbb{N} \right\}$

$S_1' = \{-1\}$

2) $S_2 = \left\{ (-1)^n + \frac{1}{n} ; n \in \mathbb{N} \right\}$

$S_2' = \{\pm 1\}$

3) $S_3 = \left\{ (-1)^m + \frac{1}{n} ; n, m \in \mathbb{N} \right\}$

$S_3' = \{\pm 1\}$

4) $S_4 = \left\{ (-1)^n + \frac{(-1)^n}{m} ; n, m \in \mathbb{N} \right\}$

$S_4' = \{+1\}$

NET 5) $S_5 = \left\{ \frac{1}{n} + \frac{1}{m} ; n, m \in \mathbb{N} \right\}$

$S_5' = \left\{ \frac{1}{n} ; n \in \mathbb{N} \right\} \cup \{0\}$

6) $S_6 = \left\{ n + \frac{1}{n} ; n \in \mathbb{N} \right\}$

$S_6' = \phi$

7) $S_7 = \left\{ \frac{1}{2^n} + n ; n \in \mathbb{N} \right\}$

$S_7' = \phi$

8) $S_8 = \left\{ \frac{1}{2^n} + \frac{1}{3^m} ; n, m \in \mathbb{N} \right\}$

$S_8' = \left\{ \frac{1}{2^n} ; n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{3^m} ; m \in \mathbb{N} \right\} \cup \{0\}$.

9) $S_9 = \left\{ n + \frac{(-1)^m}{n} ; n, m \in \mathbb{N} \right\}$

$S_9' = \phi$

10) $S_{10} = \{ n \sin n ; n \in \mathbb{N} \}$

$\{ \sin 1, 2\sin 2, 3\sin 3, \text{----} \}$ ↑ seq^n

$S_{10}' = \phi$

NET 11) $S_{11} = \left\{ \frac{1}{n} \sin \frac{1}{n} ; n \in \mathbb{N} \right\}$

$S_{11}' = \{0\}$

12) $S_{12} = \left\{ n \sin \frac{1}{n} , n \in \mathbb{N} \right\}$

$S_{12}' = \{1\}$. $\left\{ \because \lim_{n \to \infty} \frac{\sin 1/n}{1/n} = 1 \right.$

13) $S_{13} = \left\{ \frac{1}{n} \sin n , n \in \mathbb{N} \right\}$

$S_{13}' = \{0\}$ $\left\{ \because \lim_{n \to \infty} \frac{1}{n} \sin n = 0 \right.$

14) $S_{14} = \left\{ x + \frac{1}{n} ; \begin{array}{l} x \in (0,1) \\ n \in \mathbb{N} \end{array} \right\}$

$S_{14}' = [0,1]$.

Date

15) $S_{15} = \{x ; \sin x = 0\}$
$= \{0, \pi, 2\pi, 3\pi, 4\pi, ----\}$

$S_{15}' = \Phi$

16) $S_{16} = \{x : \sin x = 1\}$

$= \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ----\right\}$

$S_{16}' = \Phi$

17) $S_{17} = \{x : \cos x \neq 0\}$
$S_{17}' = [-1,1]$ if $x \in S_{18}'$?

18) $S_{18} = \{\sin n ; n \in \mathbb{N}\}$

$S_{18}' = [-1,1]$.

19) $S_{19} = \{\cos n ; n \in \mathbb{N}\}$

$S_{19}' = [-1,1]$.

20) $S_{20} = \{x \in \mathbb{R} ; x^2 + x - 2 > 0\}$

$S_{20}' = (-\infty, -2] \cup [1, \infty)$

21) $S_{21} = \{x \in \mathbb{R} ; x^2 - 4x + 3 < 0\}$

$S_{21}' = [1,3]$.

22) $S_{22} = \left\{(-1)^n + \frac{m}{n} ; n \in \mathbb{N}\right\}$

$S_{22}' = \{\pm 1\}$.

# Cantor set is unc'ble

# Perfect set is unc'ble except empty.

# A is open set then it sup and inf does Not belong to set.

# if A is bdd and closed set then it sup and inf belong to the set.

# set of Real Number is closed But Not Bdd.

# Boundary point of Rational are Real.

# Every Non empty finite set can Never open.

# $A = \mathbb{R} - \mathbb{Z}$

It is open set. But Not closed

$A' = \mathbb{R}$

$A^\circ = \mathbb{R} - \mathbb{Z}$

$A^c = \mathbb{Z}$

$(A^c)^\circ = \phi$

$\bar{A} = A \cup A' = \mathbb{R}$

$\Rightarrow$ A is dense in $\mathbb{R}$

Dense in itself:

No Nowhere Dense set

$\mathbb{R} - \mathbb{Z} \subset \mathbb{R}$ and $\mathbb{R} \not\subset \mathbb{R} - \mathbb{Z}$ so it is Not a Perfect set.

Compact set:- A Non-empty subset S of $\mathbb{R}$ is said to be compact if it is closed and Bounded.

Results :- 1) Every finite set being closed and Bounded is a compact set.

2) A set is compact iff every infinite subset of the set has a limit point in the set.

3) Sum of two compact set is compact.

4) Cartesian Product of two compact set is compact.

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Questions

Dec-2009.

Q.1. Let A and B be finite sets of m and n elements. Respectively with $m<n$ then

1) Bijective from A onto B
2) Surjective function from A onto B
3) 1-1 function from A to B and a surjective function B to A
4) Surjective function from A to B and an injective from B to A.

Soln:- $f: A \longrightarrow B$.

$|A| = m$ and $|B| = n$.

$m < n$.

then one-one Not onto.

$f: B \longrightarrow A$.

$|A| = m$ and $|B| = n$

$n > m$.

then onto Not one-one.

So option (3) is correct.

June-2011

Q.2. Let $A = \{x^2 ; 0<x<1\}$ and

$B = \{x^3 ; 1<x<2\}$

then which of the following is true.

1) There is one-one onto function from A to B
2) There is No 1-1 & onto function from $A \rightarrow B$

3) There is No 1-1 function from A to B which is onto.

4) There is No onto function from A to B which is one-one.

Soln:- Here A is open interval and B is open interval.

"Any two open interval are similar".

So $\exists$ a bijective map from A to B.

So option (1) is correct.

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Jam-2008 Qus-3 The set $U = \{x \in \mathbb{R}, \sin x = \frac{1}{2}\}$

1) open
2) Closed
3) Both open and closed
4) Neither open Nor closed

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$U = \{x \in \mathbb{R} ; \sin x = \frac{1}{2}\}$

$U = \{\frac{\pi}{6}, \frac{5\pi}{6}, \quad\}$

$D(U) = \phi$

$\bar{U} = U \cup U'$

$U \cup \phi = U$

$\Rightarrow \bar{U} = U \Rightarrow U$ is Closed

$\Rightarrow$ option (2) is correct.

Q4-23 Which of the following Subset of $\mathbb{R}^2$ are compact

a) $\{(x,y) ; |x| \le 1, |y| \le 2\}$

b) $\{(x,y) ; |x| \le 1, |y|^2 < 2\}$

c) $\{(x,y) ; x^2 + 3y^2 \le 5\}$

d) $\{(x,y) ; x^2 \le y^2 + 5\}$

(a) $\{(x,y) ; |x| \le 1, |y| \le 2\}$ closed and bdd.

(b) $\{(x,y) ; |x| \le 1, |y|^2 \le 2\}$ Closed and bdd.

(c) $\{(x,y) ; x^2 + 3y^2 \le 5\}$

$$\left\{(x,y) ; \frac{x^2}{5} + \frac{y^2}{5/3} \le 1\right\}$$

(c) is ellipise and Hence closed and bounded.

d) $\{(x,y) ; x^2 \le y^2 + 5\}$.

$$\left\{(x,y) : \frac{x^2}{5} - \frac{y^2}{5} \le 1\right\}$$

is closed but Not bdd $\Rightarrow$ Not compact.

Q4-24 Consider the set $X = (-\infty, 0) \cup \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \subseteq \mathbb{R}$ with the subspace topology. Then.

a) 0 is an isolated point

b) $(-2, 0]$ is an open set

c) 0 is limit point of the subset $\left\{\frac{1}{n}, n \in \mathbb{N}\right\}$

d) $(-2, 0)$ is an open set

(a) is incorrect.

(b) is incorrect $[-2, 0]$ is Semi closed or Semi open interval.

(c) & (d) are correct.

Ques →5 Consider the following Statements :

i) The Set of all finite Subset of the Natural Number is countable.

ii) The set of all Polynomial with Integral coefficient is countable.

Choose the Correct Statement.

a) only (i) is true.

b) only (ii) is true.

c) Both (i) and (ii) is true.

d) Both are false.

Solⁿ :- $p(x) = \{a_0 + a_1x + a_2x^2 + \text{---}\}$ $a_i \in \mathbb{Z}$ are c'ble.

Ques →6 - Match the following List I with II and Choose correct statement :-

List I

A) Countable

B) Uncountable

C) Empty set

List II

1) Set of Transcendental Element of $\mathbb{R}$ over $\mathbb{Q}$.

2) set of all functions from $Z_2 = \{0, 1\}$ to $\mathbb{N}$

3) $\{n \in \mathbb{N} ; \sqrt{n+1} - \sqrt{n}$ is Rational$\}$

Code =>	A	B	C
(a)	2	3	1
(b)	1	2	3
(c)	3	2	1
(d) ✓	2	1	3

1) Set of transcendental element of IR over Q are unc^{t}ble.

2) Set of all functions from $Z_2 = \{0,1\} \to \mathbb{N}$

$$\{ f : f : \{0,1\} \to \mathbb{N} \}$$

$$\aleph_0^2 = c^{t}\text{ble}.$$

=> (2) is c^{t}ble.

3) $\{ n \in \mathbb{N} ; \sqrt{n+1} - \sqrt{n}$ is Rational$\}$.

=> ϕ.

So option (d) is correct.

(Q-7) Which of the following is/are true?

a) Set $\{ e^{x} ; x \in \mathbb{R} \}$ is c^{t}ble set.

b) Set $\{ \log x ; x > 0 \}$ is c^{t}ble set.

c) $\{ \sin x ; -\frac{\pi}{2} < x < \frac{\pi}{2} \}$ is c^{t}ble set.

d) ✓ The set of solution of all the equation $z^n = 1$, $n = 1, 2, 3, ---$ is c^{t}ble set

option (d) is correct.

Q-18 Consider the following statements :-

Assertion (A) : ∃ a Bijection b/w $\mathbb{N}$ and set of Integer. o/r

Reason (R) : $\mathbb{N}$ is a Proper Subset of $\mathbb{Z}$.

a) Both A and R are true and R is the correct explanation of A.

b) Both A and R are true but R is Not correct explanation of A.

c) A is true but R is false.

d) R is true but A is false.

Every (countably) infinite set can be similar to its Proper Subset

Q-19 Select the correct code / Statement :-

1) Every countable set is similar to $\mathbb{N}$.

2) The set of all Disjoint intervals is not similar to the set of all real number $\mathbb{R}$.

3) The Power set of $\mathbb{N}$ is similar to the set of Real Number.

Code =>

a) 1 and 2 only.

b) 2 and 3 only

c) 1 and 3 only

d) All of these.

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Q→18 If f is function with Domain A and Range B then which of the following is true

a> B is c'ble ⇒ A is c'ble
b> A is c'ble ⇒ B is c'ble
c> A unc'ble ⇒ B is c'ble
d> None of these.

$f : \mathbb{R} \longrightarrow \mathbb{R}$.
$f(x) = 2$.
Here $\mathbb{R}$ is unc'ble. but $f(\mathbb{R}) = \{2\}$ is c'ble.

Q→18 1> The set of straight lines in a plane each of which passes through (atleast) two different points with Rational coordinate is c'ble.

2> The family of all subset of a c'ble set is c'ble

which of the following is true
a> Both
b> Neither 1 Nor 2 is correct
c> 1 is correct 2 is incorrect
d> 2 is correct 1 is incorrect

Qu→19 A be the subset of Euclidean plane $\mathbb{R}^2$ defined by $A = \{(x,y), x \geq 0, y \leq 1\}$
and $B = \{(x,y)\ 0 \leq x \leq 1, y = 0\}$

a> A and B are Not equivalent as ∄ 1-1 function from A to B
b> A and B are Not equivalent as ∄ onto function from A to B
c> A and B are Not equivalent as there is Neither one-one Nor onto Map.

d) None of these.

Solⁿ. $A = \{(x,y);\ x \geq 0,\ y \leq 1$

$A = [0,1]$ uncᵗble.

$B = \{(x,y);\ 0 \leq x < 1,\ y = 0\}$

$B = [0,1)$

Both are interval.

Any two closed, open, semi closed, semi open intervals are always equivalent.

So option (d) is correct.

Q→13 $\mathbb{N}$ be the set of Natural Number and X be any set.

1) $f: \mathbb{N} \to X$ is onto $\Leftrightarrow$ X is cᵗble.

2) X is cᵗble $\Leftrightarrow$ $f: X \to \mathbb{N}$ is one-one.

a) only 1
b) only 2
c) Neither 1 nor 2
d) Both 1 and 2

$f: \mathbb{N} \to X$ is onto.

$X \subset \mathbb{N}$

and $\mathbb{N}$ is cᵗble

subset of cᵗble set is cᵗble

$\Rightarrow X$ is cᵗble.

Now let X is cᵗble.

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$f : \mathbb{N} \to X$ is onto.

So (1) is true.

Similarly :- ∵ X is c'ble.

$f : X \to \mathbb{N}$ is one-on

if $f : X \to \mathbb{N}$ is one-one.

$X \subseteq \mathbb{N}$

and $\mathbb{N}$ is c'ble

$\Rightarrow$ X is c'ble.

$\Rightarrow f : X \to \mathbb{N}$ is one-one $\Rightarrow$ X is c'ble.

So (2) is correct.

$\Rightarrow$ Both 1 and 2 are true.

Qu→14→ Let $A = (0,1)$ and B is the set of all Disjoint Subinterval of A then

a) A and B are similar

b) ∄ onto map from A to B ~~Hence~~ hence A and B are Not similar.

c) ∃ onto map from B to A Hence they are Not similar.

d) ∄ 1-1 map from A to B hence they are Not similar.

Q→15 $f : \mathbb{Z} \to \mathbb{Z}$ s.t $f(s) = as + b$, $a, b \in \mathbb{Z}$ then f will be Identity map iff

a) $a = 1$
b) $b = 0$
c) $a = 1$, $b = -1$
d) $a = 1$ ~~or~~ ~~$a \neq 1$~~ and $b = 0$

$f(S) = aS + b$
$f(S) = S$ identity map
possible if $a = 1$, $b = 0$
=> option (d) is correct

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Q.) 18 For function f and g consider the following statement:

i) None of fog and gof may be defined.
ii) Only one of fog and gof may be defined
iii) fog and gof may both defined but may Not be equal
iv) fog and gof may be both defined and equal

Correct statement is

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a) one
b) two
c) three
d) Four

Date 28/3

Q-17 If F be the set of all function defined on $= \{1,2,3,\ldots,n\}$, $n \in \mathbb{N}$ with Range $B \subseteq \mathbb{Z}^{+}$ then

✓ a> F is c^{d}ble
b> F is unc^{d}ble
c> F is infinite
d> F is c^{d}ble if B is finite

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Let $A = \{1,2,3,\ldots,n\}$.

$F = \{f : f : A \to B\}$.

$|F| = |B|^{|A|}$

if $B = \mathbb{Z}^{+}$

then $|F| = \aleph_0^{\,n} = \underbrace{\aleph_0 \times \aleph_0 \times \cdots \times \aleph_0}_{n\text{ times}} = \aleph_0 = c^{d}$ble

$\Rightarrow$ F is c^{d}ble.

Q-18 If $A = \{(x,y);\ y = e^{x}\ ;\ x \in \mathbb{R}\}$ and
$B = \{(x,y);\ y = e^{-x}\ ;\ x \in \mathbb{R}\}$.

a> $A \cap B = \phi$
~~b>~~ $A \cap B \neq \phi$
c> $A \cup B = \mathbb{R}$
d> None of these.

$A \cap B \neq \phi$

Qu-19 $S = \{a \in \mathbb{R}\ ;\ a$ is a recurring decimal No.$\}$
$T = \{b \in \mathbb{R}\ ;\ b = \sqrt{p/q}$ for some distinct prime $p\ \&\ q\}$
then:

a> S is c^{d}ble but T is Not
~~b>~~ T is c^{d}ble but S is Not
~~c>~~ S and T both are c^{d}ble sets.
d> S and T are unc^{d}ble sets.

Q-120 which set is countable.

a) The set of all Polynomial.
b) The set of all Subset of a countable Infinite Set
c) The set A − B where A is unc'ble but B is c'ble.
d) The set of all finite subsets of $\mathbb{N}$.

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Q-120 Let A an infinite set of Disjoint Intervals of (0,1) and B be the Power set of A then

a) A and B are cardinal equal
b) A is similar to (0,1)
c) B is similar to (0,1)
d) A and B both are unc'ble.

Qus-122 Let A be the set of lines passing through the origin and slop is Integal multiple of $\pi/12$ then

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a) A is similar to $\mathbb{R}$
b) A is c'ble infinite
c) A is similar to the set of months in year.
d) A is similar to the Power set of $\mathbb{R}$.

equation of line $y = mx$. $m \in \mathbb{R}$.

$\left\{ y = \frac{\pi}{12} x \right\}$

$\left\{ \frac{\pi \cdot x}{12} \right\}$ c'ble

$\left\{ \pi \cdot \alpha \;\; \alpha \in \mathbb{Q} \right\} =$ c'ble.

Q-128 Let $f : X \to Y$ and $A, B \subseteq X$ then

a> A and B are disjoint then f(A) and f(B) are disjoint.

✓b> if f(A) and f(B) are disjoint then A and B are disjoint.

c> If A is complement of B then f(A) is a complement of f(B)

✓d> If f(A) is a complement of f(B) then A is complement of B.

A & B are disjoint then f(A) & f(B) are disjoint if f is one-one. But if f is onto then it may be disjoint or Not.

(ii) $f(A) \cap f(B) = \phi$. then $A \cap B = \phi$

Let $A \cap B \neq \phi \Rightarrow x \in A \cap B \Rightarrow x \in A \,\&\, x \in B \Rightarrow x \in f(A) \,\&\, x \in f(B)$

$x \in f(A) \cap f(B)$ ⌗

$\Rightarrow A \cap B = \phi \Rightarrow$ (b) is correct. & similarly (d) is correct.

Q→29 $f : Q \to Z$ s.t $f\left(\frac{m}{n}\right) = m$ then f is

a> one-one but Not onto

✓b> onto But Not one-one

c> Bijective

d> Not a function.

$$f : Q \Rightarrow Z$$

$$f\left(\frac{m}{n}\right) = m$$

$$f\left(\frac{1}{3}\right) = f\left(\frac{1}{4}\right)$$

$$\frac{1}{3} \neq \frac{1}{4}$$

$\Rightarrow$ f is Not one-one.

But f is onto.

Date

Q-25 $f : \mathbb{R} \to S$ and $X, Y \subseteq \mathbb{R}$ and $A, B \subseteq S$ then which one/more is true.

a) $X \supset f^{-1}(f(X))$

✓ b) $A \subset f(f^{-1}(A))$

× c) $f(X \cap Y) = f(X) \cap f(Y)$

✓ d) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

e) $f(A_1 - A_2) \subseteq f(A_1) - f(A_2)$

$f(X \cap Y) \subseteq f(X) \cap f(Y)$

Q-26 For Non-Negative Integer m, n defines a function as follows.

$$f(m,n) = \begin{cases} n+1 \; ; & \text{if } m=0 \\ f(m-1,1) \; ; & \text{if } m \neq 0, n=0 \\ f(m, n-1) \; ; & \text{if } m \neq 0, n \neq 0 \end{cases}$$

then the value of $f(1,1)$ is

a) 4

b) 3

✓ c) 2

d) 1

$f(1,1) = f(m-1,1) = f(0,1) = n+1 = 1+1 = 2.$

Q-27 let $A = \{1,2,3,4\}$ which of the following is Not a function from A to A.

a) $\{(1,4), (4,1), (1,2), (4,3)\}$

b) $\{(4,1), (1,2), (2,3), (3,4)\}$

c) $\{(1,4), (2,1), (3,2), (4,1)\}$
d) $\{(4,1), (1,4), (2,2), (3,3)\}$.

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Q-128 If $\alpha = \frac{1+\sqrt{5}}{3}$, $\beta = \frac{\pi}{8!}$ then

a) α is Algebraic but β is Not.
b) β is Algebraic but α is Not
c) Both α and β are Algebraic
d) Neither α nor β is Algebraic.

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Q-129 Let X denote the two point set $\{0,1\}$ and $X_j = \{0,1\}\ \forall\ j = 1,2,3,\ldots$ Let $Y = \prod_{j=1}^{\infty} X_j$
which of the following is/are true.

a) Y is c'ble set
b) $|Y| = \text{card}\,[0,1]$
c) Y is unc'ble
d) $\bigcup_{n=1}^{\infty} \left(\prod_{j=1}^{n} x_j\right)$ is unc'ble.

Q-130 :- The Number $\sqrt{2}\,e^{i\pi}$ is a

a) Rational Number
b) Transcendental No.
c) Irrational Number.
d) Imaginary Number.

Q-31 Consider the two set $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$ choose the correct statement

1) Total maps from A to B is 125 ✓
2) Total maps " A to B is 243
3) Total one-one map from A to B is 60 ✓
4) Total one-one map from A to [illegible]

~~Total~~ map $A \to B$

$|B|^{|A|} = 5^3 = 125$

Total one-one Map from $A \to B$

$^nC_m \cdot m!$ $\qquad {}^nP_m$

$5C_3 \cdot 3!$

$\frac{5 \times 4 \times 3}{1 \times 2 \times 3} \times 3! = 60$

Q-32 Let $E = \{z \in \mathbb{C} ; e^z = i\}$ then E is

a) a singlton
b) E is a set of 4 element
c) E is an infinite set ✓
d) E is an infinite group under addition

Q-33 $A = \{$[illegible]$; m, n \in \mathbb{Z}\}$

a) A is dense in $\mathbb{R}$
b) A has only c'bly many limit point in $\mathbb{R}$.
c) A has no limit point in $\mathbb{R}$
d) only Irrational No. can be limit point of A

$\overline{A} = \mathbb{R} \Rightarrow A$ is dense in $\mathbb{R}$.

Date 14

Q-34. Let C be the Standard Cantor's middle third Set. Then

a) C is Not Measurable

b) C is countable and measure zero.

c) C is unc'ble and of measure zero.

d) C is unc'ble and of measure positive

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June
Q→35 The set $\left\{ \frac{1}{n} \sin\frac{1}{n} ; n \in \mathbb{N} \right\}$

1) one limit point is 0

2) one limit point is -1

3) three limit point $\{1, 0, -1\}$

4) One is limit point is 1

$\lim_{n\to\infty} \frac{1}{n} \sin\frac{1}{n} = 0$ = limit ⇒ limit point.

2009
Q→36. Lim sup $\left\{ \frac{(-1)^n}{2^n} ; n \in \mathbb{N} \right\}$

1) 0

2) 1/4

3) 1

4) -1

$S = \left\{ \frac{(-1)^n}{2^n} ; n \in \mathbb{N} \right\}$

$\text{lim sup} = \left\{ \frac{(-1)^n}{2^n} ; n \in \mathbb{N} \right\}$

$\inf \sup \left\{ \frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots \right\}$

$\inf \{0, 0, 0, \ldots\}$

$= 0$

$\Rightarrow \lim\sup S = 0$

June-2010.

Q→37 Let A and B be finite sets of m and n elements. Then No. of function $f: A \to B$ is

1> m^n

2> $m+n$

3> mn

4> ✓ n^m

Jam-2006.

Qus→38 Set of all Boundary point of Q in IR is

1> ✓ IR

2> IR/Q

3> Q

4> Φ

June-2010

Qus→39 Let S be infinite Subset of IR s.t $S \cap Q = \Phi$, which of the following is true.

1> S must have a limit point which belong to Q.

2> S must have a limit point which belong to R/Q

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3) S cannot be a closed set in $\mathbb{R}$.

4) R/S must have a limit point which is belong to S.

$S_1 = \{\sqrt{2}+\frac{1}{n} ; n \in \mathbb{N}\}$

$S \cap \mathbb{Q} = \phi$

$\sqrt{2}$ is limit point of S But Not belong to $\mathbb{Q}$.

$\Rightarrow$ (1) is incorrect.

$S = \{\frac{1}{\sqrt{p}} , p$ is prime and $p \in \mathbb{N}\}$

limit point of S is 0 and $0 \notin \mathbb{R}-\mathbb{Q}$.

$\Rightarrow$ (2) is incorrect.

$S = \{\sqrt{p} ;$ p is prime $p \in \mathbb{N}\}$

$S' = \phi$

$\overline{S} = S$

$\Rightarrow$ S is closed

$\Rightarrow$ (3) is incorrect.

$S \cap \mathbb{Q} = \phi$

$S \subset \mathbb{R} - \mathbb{Q}$

$\mathbb{Q} \subset \mathbb{R} - S$

$D(\mathbb{Q}) \subset D(R-S)$

$D(\mathbb{Q}) = R$.

$S \subset R = D(\mathbb{Q}) \subset D(R-S)$

Complete Set S is limit point of $\mathbb{R}-S$.

i.e Every element of set S is limit point of $\mathbb{R}-S$

i.e R/S must have a limit point which is belong to set S.

$\Rightarrow$ (4) is correct.

Ous→40 find derived set of $S = \left\{ \frac{1}{x} ; x \in (0,1] \right\}$

$S = \{1, 10, 20, 30, \dots\}$

$S' = [1, \infty)$

and S contains all its limit point.

i.e S is closed.

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Jam-2011

Ous→41 Consider the following subset of IR.

$E = \left\{ \frac{n}{n+1} ; n \in \mathbb{N} \right\}$ and

$F = \left\{ \frac{1}{1-x} ; 0 \le x < 1 \right\}$

a) Both E and F closed
b) E closed and F is Not closed
[✓ c)] E is Not closed and F is closed.
d) Neither E is closed Nor F is closed.

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Limit point of E is 1 and $1 \notin E$

⇒ E is Not closed

and $F = \left\{ \frac{1}{1-x} ; 0 \le x < 1 \right\}$

$F' = [1, \infty)$

F contains all of its limit points.

⇒ F is closed.

IISC-2008

Q→42 Let $X = \left\{ \frac{1}{n} ; n \in \mathbb{N} \right\}$ and $\bar{X}$ be its closure

Then.

a) $\bar{X}/X$ is a single point.

b) $\bar{X}/X$ is open in $\mathbb{R}$

c) $\bar{X}/X$ is infinite But Not open in $\mathbb{R}$.

d) $\bar{X}/X = \phi$

$\bar{X} = X \cup \{0\}$

$\bar{X}/X = \{0\}$ is a single point.

IISC

Q→43 Let $A \neq R$ be dense in $\mathbb{R}$ if $U \subseteq \mathbb{R}$ is Non-empty open Subset then.

i) $U \subseteq \overline{A \cap U}$

ii) $\overline{A \cap U} = \phi$

iii) $\overline{A \cap U} \subseteq U$

iv) $\overline{A \cap U} = A \cap \bar{U}$

$A = \mathbb{R} - \{2\}$

$U = (2,3) \subseteq \mathbb{R}$

$\bar{U} = [2,3]$

$A \cap U = (\mathbb{R} - \{2\}) \cap (2,3)$

$= (2,3)$

$\overline{A \cap U} = [2,3]$

$A \cap \bar{U} = (\mathbb{R} - \{2\}) \cap [2,3]$
$= (2,3]$

$\overline{A \cap U} \neq A \cap \bar{U}$
So option (i) is correct.
$(2,3) \subseteq [2,3]$

$\overline{A \cap U} = [2,3] \neq \phi$
So option 2nd is incorrect.

$[2,3] \not\subseteq (2,3)$ So option (iii) is incorrect.

NBHM
Q→44. The set $\{x \in \mathbb{R} ; x^2 > x\}$ is same as.

1) The Interval (0,0)
2) The complement of the interval (0,1).
[✓ 3) The complement of the interval [0,1].]
4) The interval [0,1]

MBHM
Qus→45. The set $\{x \in \mathbb{R} : |x+1| = |x|+1\}$ is same as

1) $\{x \in \mathbb{R} ; x > 0\}$
[✓ 2) $\{x \in \mathbb{R} ; x \geq 0\}$]
3) Whole $\mathbb{R}$
4) None of the above.

Qus→46 The set $\{x \in \mathbb{R} : [x+1] = [x]+1\}$ is same as.

Whole $\mathbb{R}$.

NBHM 2004

Qus 47 How many subsets A of $\{1,2,3,4\}$ are there that $(A\cup\{1,2\})-(A\cap\{1,2\})$ has exactly one point.

1> 1
2> 2
3> 3
[✓ 4> 4]

Qus ⑱ Find derived set of $\left\{\frac{x}{1+x}\ ;\ x\in\mathbb{R}\right\}-\{-1\}$.

Qus ㊾ Find derived Set of $\left\{\frac{x}{1+|x|}\ ;\ x\in\mathbb{R}\right\}$

2006
Qus 50 The set $\{x\in\mathbb{R}\ ;\ x\sin x\le 1,\ x\cos x\le 1\}\subseteq\mathbb{R}$

1> Bdd and closed
2> Bdd and open
[✓ 3> unbdd and closed]
4> unbdd and open.

TIFR-2009
Qus 51 Let f be an 1-1 function from closed interval $[0,1]$ to the set of Real No $\mathbb{R}$ then

a> f must be onto
b> Range of f must contain a Rational Number.

c) Range of f must contain an irrational No

d) Range of f must contain both Rational & irrational

$f : [0,1] \longrightarrow \mathbb{R}$

$f(x) = x$

this is Not onto

$f : [0,1] \longrightarrow \mathbb{R}$

$f(x) = x + \sqrt{2}$

f is one-one Must contains irrational No.

2009

I.I.Sc

Q(ii)→52 How many complex Number $z = x + iy$ are there s.t $x + y = 1$ and $\exp i(x^2+y^2) = 1$

a) zero

b) Non-zero but finitely many

c) c'ble.

d) unc'ble.

$e^{i(x^2+y^2)} = 1$

$|\cos(x^2+y^2) + i\sin(x^2+y^2)| = |1|$

$\cos^2(x^2+y^2) + \sin^2(x^2+y^2) = 1$

and $x + y = 1$

$y = 1 - x$

i.e completely $\mathbb{R}$ which satisfied $\cos^2(x^2+y^2) + \sin^2(x^2+y^2) = 1$

$\Rightarrow$ unc'ble.

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IISc 2010

Qus 53 Let S be a collection of pairwise disjoint open sets in the plane $\mathbb{R}^2$, Then

1> S cannot be finite.
2> S cannot be c'ble
[✓ 3>] S cannot be unc'ble
4> S must be empty.

Pairwise disjoint open interval are c'ble.

MBHM 2004

Qus 54 Which of the subsets of the plane is bdd set.

[✓ 1>] $\{(x,y) ; x^2+y^2<10\}$
2> $\{(x,y) ; 1<\frac{x^2}{4}+\frac{y^2}{9}\}$
3> $\{(x,y) ; x=y\}$
4> $\{(x,y) ; y^2<4x\}$

NBHM

Qu 4-55 Which of the following sets satisfy the condition that for every +ve integer n there is some a in A s.t $a<n$

[✓ a>] $A = \{-1, 5\}$
b> $A = \phi$
c> $A = \mathbb{N}$
d> $A = \{x \in \mathbb{R} ; x > 10\}$

NBHM 2006
Que-56 If $A = \{E \subset \mathbb{N} / E$ is disjoint from the set of all multiples of $3\}$ where $\mathbb{N}$ is set of all +ve integers then which of the following is always true

1> $\{6\} \in A$
2> $6 \in E$
3> $6 \in A$
4> if $6 \in E$, then E does Not belong to A.

TIFR-2008
Que 57 Let S be set of all seqⁿ $\{a_1, a_2, \ldots, a_n, \ldots\}$ where each entry a_i is either 0 or 1 then S is c'ble.

$f : \mathbb{N} \to \{0, 1\}$.
$2^{\aleph_0} = c$ unc'ble.

Que-58 Let $P(x) = \{x^n + 5x^2 + 7x,\ n$ odd, $n \geq 3\}$ fix $a \in \mathbb{R}$
$S = \{x \in \mathbb{R} ; P(x) = a\}$ then

1> $S = \phi$
2> S is finite and Non-empty.
3> S is c'ble
4> S is unc'ble.

$P(x) - a = 0$
$x^n + 5x^2 + 7x - a = 0$
Depends upon n $n \in \mathbb{N}$
i.e countably infinite.

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NET
Que-59 Let X denote the two point set $\{0,1\}$ and write $X_j = \{0,1\}$ for every $j = 1,2,3, ----$ Let $Y = \prod_{j=1}^{\infty} X_j$ which of the following is/are true

a) Y is a c'ble
b) card Y = card [0,1]
c) $U\left(\prod_{j=1}^{n} X_j\right)$ is unc'ble
d) Y is unc'ble

NET
Que-60 For each $j = 1,2,3, ----$ Let A_j be finite set containg at least two distinct element then

a) $\bigcup_{j=1}^{\infty} A_j$ is c'ble
b) $\bigcup_{n=1}^{\infty} \prod_{j=1}^{n} A_j$ is unc'ble
c) $\prod_{j=1}^{\infty} A_j$ is unc'ble
d) $\bigcup_{j=1}^{\infty} A_j$ is unc'ble.

Dec-2013.

Qu-6) Which of the following subset of $\mathbb{R}^2$ is unc^{t}ble.

a) $\{(a,b) \in \mathbb{R}^2 \,;\, a \leq b\}$
b) $\{(a,b) \in \mathbb{R}^2 \,;\, a+b \in \mathbb{Q}\}$
c) $\{(a,b) \in \mathbb{R}^2 \,;\, ab \in \mathbb{Z}'\}$
d) $\{(a,b) \in \mathbb{R}^2 \,;\, a,b \in \mathbb{Q}\}$

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2014 June.

Qu-6) Let W is set of all constant function on $\mathbb{R}$
X is set of all Poly on $\mathbb{R}$
Y is set of all cts fun on $\mathbb{R}$
Z is set of all constant f^n on $\mathbb{R}$.

Which of these has the same cardinality as that of $\mathbb{R}$.

a) only W
b) W and X
c) W, X, Z
d) All X, Y, Z, W

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SEQUENCE OF REAL NUMBERS

Sequence :- Let S be a set and IN be the Set of all Natural Number. Then A sequence is a function whose Domain is IN and Range is S

i.e A sequence in a set S is a rule which assigns to each Natural Number to unique element of S.

Real sequence or sequence of Real Number :-

Sequence is a function IN to IR is called seq^n of Real Number.

$$S = \{ f : f : \mathbb{N} \longrightarrow \mathbb{R} \}.$$

terms of the sequence are the image of the functio

i.e $\langle f(1), f(2), f(3), - - - - - \rangle$

$f(n)$ is the n^{th} term of the sequence.

$$f : \mathbb{N} \longrightarrow \mathbb{R}.$$

seq^n can be Represented by $\{f_n\}$, $\langle f_n \rangle$, and $\langle f_n \rangle$

or. $\{ f(1), f(2), f(3), f(4), - - -, f(n), - - - \}$

$\{ a_1, a_2, a_3, a_4, - - - - -, a_n - - - - \}$

Hence we can adop $\boxed{\langle a_n \rangle = \langle f(n) \rangle}$

Note :- The Number of ~~elem~~ terms in a sequence is always infinite

Note :- We define a sequence as an ordered set of Real Numbers whose members can be put in an one-one correspondence with set of Natural Number. However, a sequence may have only a finite Number of distinct elements.

Example of sequences of Real Numbers

1> $a_n = \langle n \rangle = \langle 1, 2, 3, \ldots \rangle$

2> $a_n = \langle -n \rangle = \langle -1, -2, -3, \ldots \rangle$

3> $a_n = \langle n^2 \rangle = \langle 1^2, 2^2, 3^2, \ldots \rangle$

4> $a_n = \langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$

5> $a_n = \langle 1 + \frac{1}{n} \rangle = \langle 2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots \rangle$

6> $a_n = \langle \sin n\pi \rangle = \langle 0 \rangle = \langle 0, 0, 0, \ldots \rangle$

7> $a_n = \langle (1)^n \rangle = \langle 1, 1, 1, \ldots \rangle$

8> $a_n = \langle 2^n \rangle = \langle 2, 2^2, 2^3, \ldots \rangle$

9> $a_n = \langle n^{(-1)^n} \rangle = \langle 1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots \rangle$

10> $a_n = \langle \frac{-n}{2} \rangle = \langle \frac{-1}{2}, \frac{-2}{2}, \frac{-3}{2}, \frac{-4}{2}, \ldots \rangle$

11> $a_n = \langle \cos(2n+1)\frac{\pi}{2} \rangle = \langle 0 \rangle = \langle 0, 0, \ldots \rangle$

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12) $a_n = \begin{cases} 0, & n \text{ is } 1 \text{ or prime} \\ n, & \text{else} \end{cases}$

$= \langle 0, 0, 0, 4, 0, 6, 0, 8, 9, 10, 0$

13) $a_n = \begin{cases} 2, & n \text{ is } 1 \text{ or prime} \\ p, & p \text{ is smallest prime divisor of } n \end{cases}$

14) $a_n = \begin{cases} -1-\frac{1}{n}, & n \text{ is odd} \\ 1+\frac{1}{n}, & n \text{ is even} \end{cases}$

15) $a_{n+2} = a_n + a_{n+1}$ Here $a_1 = 1$, $a_2 = 1$.

16) $a_{n+2} = \frac{1}{2}(a_n + a_{n+1})$ given a_1, a_2

17) $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$

18) $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$

Bounded Sequence :- A sequence $\langle a_n \rangle$ is said to be Bounded iff its Range set is bounded.

or.

A seq^n $\langle f(n) \rangle$ is bdd, if $\exists$ +ve integer $M > 0$ s.t

$$|f(n)| < M, \quad n \in \mathbb{N}.$$

ie $<a_n>$ is bounded.

if $\exists$ $m, M \in \mathbb{R}$ s.t $m \leq a_n \leq M$ $\forall n \in \mathbb{N}$

~~Super~~ supremum and infimum of the Range set called Sup. and Inf. of sequence. Respectively.

Bounded above :- Let $S \neq \phi$ subset of IR s.t $x \leq M$ $\forall x \in S$ and M is upper Bound.

Bounded below :- Let S be Non-empty Subset of IR s.t $x \geq m$ $\forall$ $x \in S$ and m is lower Bound.

IIrly :- $f(x) \leq M$ → upper Bound

$f(x) \geq M$ → lower Bound.

Range set :- All distinct image of function or seq^n.

or Range set of a seq^n in which all element are distinct.

Note :- A set has infinite/uncible upper Bound and lower Bound.

Ex :- $S = \{1, 2, 3, 4\}$.

4 is upper Bound and 1 is lower Bound

$M = [4, \infty) = \{x : x \in [4, \infty)\}$ - upper Bound

$m = (-\infty, 1]$ $\{x ; x \in (-\infty, 1]\}$ - lower Bound

ie A set has infinite/uncible upper Bound and lower Bound.

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Clopen :- A set which is closed as well as open in Real line is called clopen set.
$\mathbb{R}$, ϕ is clopen set in Real line.

Note :- There is No clopen set in $\mathbb{R}$ other than ϕ ~~∞~~ and $\mathbb{R}$.

Note :- Inf and sup of a set are unique.

L.U.B → Superimum → unique.
G.L.B → Infimum → unique.

Ex :- In Natural Number $\mathbb{N}$ inf & sup is
Inf is 1, sup is Not defined
But in Extended ~~comp~~ Real line.
Inf is 1, sup is $+\infty$

Ex :- ϕ has sup and inf in Extended Real Number

g.l.b − inf → ∞
L.U.B − sup → $(-\infty)$
sup < inf

Monotonic sequence

1) Seq. $\langle a_n \rangle$ is said to be Monotonically increasing (Non decreasing) if $\forall\ n \in \mathbb{N}$; $a_n < a_{n+1}$ $[\langle a_n \rangle \uparrow]$

2) Seq. $\langle a_n \rangle$ is said to be monotonically decreasing (Non. increasing) if $\forall\ n \in \mathbb{N} \ni a_n \geq a_{n+1}$ $[\langle a_n \rangle \downarrow]$

3) Seq $\langle a_n \rangle$ is strictly increasing (increasing) if $\forall\, n \in \mathbb{N}$; $a_n < a_{n+1}$ denoted by $\langle a_n \rangle \uparrow$

4) Seq $\langle a_n \rangle$ is strictly decreasing (decreasing) if $\forall\, n \in \mathbb{N}$ $a_n > a_{n+1}$ denoted by $\langle a_n \rangle \downarrow$

Subsequence:- Let $\langle a_n \rangle = \{a_1, a_2, a_3, ---\}$ be a sequence in $\mathbb{R}$. A subsequence of a sequence whose terms are choosen from the original seqn in the same order as in original seqn.

Eg. $\{a_3, a_5, a_7, ----\}$ and $\{a_{10}, a_{15}, a_{20}, a_{22}, ---\}$ are subsequences of $\langle a_n \rangle$.

But $\{a_5, a_3, a_6, a_2 ---\}$ and $\{a_{10}, a_9, a_8, a_7, ---\}$ are Not subsequence of seqn $\langle a_n \rangle$ because the order of terms is lost.

Note:- $\{a_1, a_2, a_2, a_2, a_3, a_4, ----\}$ is Not subseq of $\langle a_n \rangle$ because here term a_2 appears more than once which should not be.

Hence we can define a sequence as.
if $\langle a_n \rangle$ be a seqn, then any infinite succession of its terms, picked out in any way (but preserving the original order) is called Subseqn of $\langle a_n \rangle$.
In Otherwords.

Let $\langle n_k \rangle$ be strictly increasing sequence of Natural Number and $\langle a_n \rangle$ be any seqn then $\langle a_{n_k} \rangle$ is called Subseqn of $\langle a_n \rangle$

i.e $f : \mathbb{N} \longrightarrow \mathbb{N}$

$f(k) = n_k$.

Eg. $\langle n_k \rangle = \langle 2k \rangle$ then $\langle a_{2k} \rangle$ is subseq^n of $\langle a$

$\langle n'_k \rangle = \langle 2k-1 \rangle$ then $\langle a_{2k-1} \rangle$ is " " $\langle a_n \rangle$

Complementry Subsequences :-

Let $\langle a_{n_k} \rangle$ and $\langle a_{n'_k} \rangle$ be two subseq^n of $\langle a_n \rangle$ sot $T_1 = \{n_k, k \in \mathbb{N}\}$ + $T_2 = \{n'_k, k \in \mathbb{N}\}$ and if $T_1 \cap T_2 = \phi$ & $T_1 \cup T_2 = \mathbb{N}$ then $\langle a_{n_k} \rangle$ and $\langle a_{n'_k} \rangle$ is called subseq^n.

Eg: $\langle a_n \rangle$ be a seq^n $\langle a_{2k} \rangle$ + $\langle a_{2k-1} \rangle$ be two subseq^n.

$T_1 = \{n_k; k \in \mathbb{N}\} = \{2k, k \in \mathbb{N}\} = \{2, 4, 6, 8, ---\}$

$T_2 = \{n'_k; k \in \mathbb{N}\} = \{2k-1, k \in \mathbb{N}\} = \{1, 3, 5, 7, ---\}$

Here $T_1 \cap T_2 = \phi$ and $T_1 \cup T_2 = \mathbb{N}$. Hence

$\langle a_{2k} \rangle = \{a_2, a_4, a_6, ---\}$ and $\langle a_{2k-1} \rangle = \{a_1, a_3, a_5, ---\}$ are two Complementry subseq^n of seq^n $\langle a_n \rangle$.

Limit points of a sequence :- Let $\langle a_n \rangle$ be a seq^n of Real Number and $l \in \mathbb{R}$, Then we say l is a limit point of seq^n $\langle a_n \rangle$ if every Nbd of l contains infinite members of the seq^n $\langle a_n \rangle$.

i.e a_n is arbitrary close to l for infinite value of

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n.
or infinitely many terms of the seq^n occure very close to l.

Note:- l is limit point of $\langle a_n \rangle$ iff for any $\epsilon > 0$; $a_n \in (l-\epsilon, l+\epsilon)$ for infinite value of n.

Note:- If l is limit point of $\langle a_n \rangle$ then every Nbd of l though contains infinite member of sequence but can leaves out infinite member of seq^n it is quite possible.

Note:- If a Real Number appears in a seq infinite many times then it is l.p but Not conversely.

Eg:- $\langle a_n \rangle = \langle (-1)^n \rangle = \{-1, 1, -1, 1, ---\}$
So 1 and -1 are limit point of $\langle a_n \rangle$

i.e if $\exists \ \alpha \in \mathbb{R}$ s.t
$a_n = \alpha$ for infinite values of n.
$\Rightarrow \alpha$ is a limit point.

Note:- if l is a limit point of Range $\langle a_n \rangle$ then l is limit point of seq^n $\langle a_n \rangle$ but Not conversely.

Eg:- $\langle a_n \rangle = \langle c \rangle$ Here $c \in \mathbb{R}$.
$\langle a_n \rangle$ called Constant seq^n.
Here 'c' is limit point of $\langle a_n \rangle$ but Not L.P of Range $\langle a_n \rangle = \{c\}$, and finite set can't have limit point.

X Note:- If $a_n \in (l-\epsilon, l+\epsilon)$ for an infinite value of n and $a_n = l$ for at the most finite Number of values of n, then a limit point l of the seqn $\langle a_n \rangle$ as well as limit point of the Range $\langle a_n \rangle$.

Example.

1) $\langle a_n \rangle = \langle \frac{1}{n} \rangle$; $n \in \mathbb{N}$.

}

has '0' as a limit point.
and $a_n = 0$ for No value of n
$\Rightarrow$ 0 is limit point of Rage $\langle a_n \rangle$.

2) $\langle a_n \rangle = \langle 1 + (-1)^n \rangle$; $n \in \mathbb{N}$
$= \{0, 2, 0, 2, 0, 2, 0, \text{-----}\}$

0 and 2 only limit points of $\langle a_n \rangle$.
But No limit point of Range set b'coz Range $\langle a_n \rangle = \{0, 2\}$ is finite. and finite set has No limit point.

3) $\langle a_n \rangle = \langle (-1)^n \cdot \left(1 + \frac{1}{n}\right) \rangle$

$= \{-2, \frac{3}{2}, \frac{-4}{3}, \frac{5}{4}, \frac{-6}{5}, \text{---}\}$

1 and -1 are limit points of $\langle a_n \rangle$ and also limit points of Range $\langle a_n \rangle$ b'coz $a_n \neq 1$ or -1 for any value of n.

Existence of limit points.

Thm- ~~Bolo~~ Bolzano – Weierstrass Theorem [for seq^n]. Every Bounded seqn has a limit point.

Note:- Converse of above thm is Not always true. ~~i.e if~~ seqn has a limit point then $\langle a_n \rangle$ will be Bounded. It is false statement.

E.g:- $\langle a_n \rangle = \{1, 2, 1, 4, 1, 6, 1, 8, \ldots\}$

1 is l.p of $\langle a_n \rangle$ but $\langle a_n \rangle$ is Not bdd.

Thm:- The set of limit points of a bdd seq^n has the greatest and the least member.

Note:- The greatest and smallest of the limit points of a Bounded sequence are called upper limit and lower limit.

E.g:- $\langle a_n \rangle = (-1)^n$, upper limit $= 1$, lower limit $= -1$

$\langle a_n \rangle = \{1 + (-1)^n\}$; upper limit $= 2$, lower limit $= 0$

$\langle a_n \rangle = \left\{\frac{(-1)^n}{n}\right\}$; Here '0' is the only limit point. So upper limit $= 0$ and lower limit $= 0$.

Limit of Sequence:- A seqn ~~of~~ $\langle a_n \rangle$ of Real Number and $l \in \mathbb{R}$, we say 'l' is the limit of $\langle a_n \rangle$ if for any $\epsilon > 0$; $\exists$ $m \in \mathbb{N}$ (depending on ϵ) s.t $|a_n - l| < \epsilon$ $\forall$ $n \geq m$

Then we write $a_n \to l$ as $n \to \infty$

or $\boxed{\lim_{n\to\infty} a_n = l}$

or $a_n \to l \quad \forall \ n \geq m$

or $\lim a_n = l \quad \forall \ n \geq m$.

Note:- Limit ⇒ limit point (for Bounded seq^n).
Limit point ⇏ limit.

Convergence of sequence:- Sequence is said to be convergence if it has limit and we say seq^n converges to that limit.

Note:- if l is a limit of $\langle a_n \rangle \Rightarrow \langle a_n \rangle$ is converges.

$\Rightarrow |a_n - l| < \epsilon \ ; \ \forall \ n \geq m$

$\Rightarrow l - \epsilon < a_n < l + \epsilon \ ; \ \forall \ n \geq m$

$\Rightarrow a_n \in (l - \epsilon, l + \epsilon) \ ; \ \forall \ n \geq m$

$\Rightarrow l$ is a limit point of $\langle a_n \rangle$.

Note:- From the above defination we can say after $(m-1)^{th}$ term, all terms of seq^n contained in open interval $(l-\epsilon, l+\epsilon)$. Hence if l is limit of $\langle a_n \rangle$ then every Nbd of l (or every open interval containing l) leaves out $(m-1)$ terms or leaves out only finite terms of seq^n.

Note:- if Any open interval containing l leaves out infinite members of seq^n. then l is Not a limit of seq^n.

Note: if $\langle a_n \rangle$ be a seq^n of Non-Negative Numbers and if $\lim_{n\to\infty} a_n = l$ then $l \geq 0$.

Note: if $\langle a_n \rangle$ be a seq^n of Real Number, if $a_n \leq M$ $\forall$ $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = l$ then $l \leq M$.

Note: if l is limit then it is only l.p of $\langle a_n \rangle$

Note:- if $\langle a_n \rangle$ has more than one limit point then it has No limit.

# Which of the following sequence convergent or Not-

1) $\langle a_n \rangle = n^2 = \{1^2, 2^2, 3^2, 4^2, ----\}$ $\Rightarrow$ Not cgt.

2) $\left\langle \frac{1}{n} \right\rangle \Rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$ $\Rightarrow$ converge to zero.

3) $\left\langle 1 + \frac{1}{n} \right\rangle \Rightarrow$ converges to 1.

4) $\langle \sin n\pi \rangle = \langle 0, 0, 0, \dots 0, -- \rangle$ - Converges to 0.

5) $\langle (1)^n \rangle = \langle 1, 1, 1, ------ \rangle$ = converges to 1.

6) $\langle 2^n \rangle$ = Not cgt.

7) $\langle n^{(-1)^n} \rangle = \{1, 2, \frac{1}{3}, 4, -----\}$

unbdd above and bdd below seq^n.
i.e Not bdd. $\Rightarrow$ Not cgt.
" Every cgt seq^n is bdd "

8) $\langle -\frac{n}{2} \rangle = \{ -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots \} \Rightarrow$ Not cgt.

9) $\langle \cos (2n+1)\frac{\pi}{2} \rangle = \{ 0, 0, 0, \ldots \} \Rightarrow$ converge to 0

Note:- Unique limit point iff limit (for bdd seq^n)

Note:- A seq^n have atmost one limit in $\mathbb{R}$.

Note:- Every cgt seq^n is bdd. But Not conversely.

Note:- Not bdd $\Rightarrow$ Not cgt.

Note:- Unbdd seq^n $\not\to$ Real Number

Note:- Limit of Range of sequence $\Rightarrow$ limit point of the seq^n But Not conversely.

Ex:- $a_n = \langle (-1)^n \rangle$

$a_n = \{ -1, 1, -1, 1, \ldots \}$.

a_n has two limit point.

$R\langle a_n \rangle = \{ -1, 1 \}$.

$R'\langle a_n \rangle = \phi$

But Range Space has no limit point.

Ex → $a_n = \langle (1)^n \rangle$

$a_n = \{ 1, 1, 1, 1, \ldots \}$

$R\langle a_n \rangle = \{1\}$.

$R'\langle a_n \rangle = \phi$

Here this sequence have limit point 1 and unique and seq^n bdd $\Rightarrow$ seq^n is cgt.

Ex:- $a_n = \langle (-1)^n + \frac{1}{n} \rangle$

$$R\langle a_n \rangle = \begin{cases} -1 + \frac{1}{n} & ; \ n \text{ is odd} \\ 1 + \frac{1}{n} & ; \ n \text{ is even} \end{cases}$$

$R' \langle a_n \rangle = \{1, -1\}$

⇒ seq^n is bdd but Not unique limit point

⇒ Not cgt.

Ex:- $\langle a_n \rangle = \langle \sin(2n+1)\frac{\pi}{2} \rangle$

$\langle a_n \rangle = \{ \sin\frac{3\pi}{2}, \sin\frac{5\pi}{2}, \sin\frac{7\pi}{2}, --- \}$

$= \{-1, 1, -1, 1, --- \}$.

$\langle a_n \rangle$ has two limit point 1 & -1

$R \langle a_n \rangle = \{1, -1\}$.

But Range of $\langle a_n \rangle$ have No limit point.

Seq^n is Not cgt.

Ex:- $\langle a_n \rangle = \langle \sin(4n+1)\frac{\pi}{2} \rangle$

$= \{ \sin\frac{5\pi}{2}, \sin\frac{9\pi}{2}, \sin\frac{13\pi}{2}, --- \}$

$= \{ \sin\frac{\pi}{2}, \sin\frac{\pi}{2}, ---- \}$

$= \{1, 1, 1, --- \}$

$R \langle a_n \rangle = \{1\}$.

Seq^n is bdd and unique limit point '1'

⇒ seq^n is cgt.

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Eg:- $\langle a_n \rangle = \langle \sin(4n-1)\frac{\pi}{2} \rangle$

$\langle a_n \rangle = \{-1, -1, - - \}$

$R\langle a_n \rangle = \{-1\}$.

seq^n is bdd and unique limit point $\Rightarrow$ seq^n is cgt.

Eg:- $\langle a_n \rangle = \langle \cos n\pi \rangle$

$\{a_n\} = \{\cos\pi, \cos 2\pi, - - - \}$

$\{-1, 1, -1, 1, - - - \}$

$R\langle a_n \rangle = \{1, -1\}$

Not cgt.

Eg:- $\langle a_n \rangle = \langle \cos 2n\pi \rangle$

$= \langle \cos 2\pi, \cos 4\pi, - - - \rangle$

$R\langle a_n \rangle = \{1\}$.

cgt an since. bdd + unique limit point is '1'.

Eg:- $\langle a_n \rangle = \langle \cos(2n+1)\frac{\pi}{2} \rangle$

$\langle a_n \rangle = \langle 0, 0, - - - - - \rangle$

$R\langle a_n \rangle = \{0\}$.

cgt to 0

since bdd + unique limit point '0'.

Eg:- $\langle a_n \rangle = \langle \cos(4n+1)\frac{\pi}{2} \rangle$

$\langle a_n \rangle = \langle 0, 0, 0, - - - - - \rangle$

$R\langle a_n \rangle = \{0\}$

Converges to 0.

Eg:- $\langle a_n \rangle = \langle \cos(2n+1)\pi \rangle$

$\langle a_n \rangle = \langle -1, -1, - - - \rangle$

$R\langle a_n \rangle = \{-1\}$

cgt to -1

Note:- Every cgt seq^n is bdd. converse of this th^m is Not true
Eg $(-1)^n$.

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Note:- A seq^n can't converge to more than one limit or Every cgt seq^n has unique limit.

Note:- A sequence can Never be unc'ble.

Note:- A seq^n is cts and differentiable.

Note:- Range set of seq^n can Never be empty.

Def^n:- Sequence Diverge to $+\infty$] div / cgt $\to \infty$.
Seq^n converge to $+\infty$

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A seq^n $\langle a_n \rangle$ is said to diverge to $+\infty$ if given $K \in \mathbb{R}$ However large, $\exists\ m \in \mathbb{N}$ (depending on K) s.t. $a_n > K\ \forall\ n \geq m$. and we write $\lim_{n\to\infty} a_n = +\infty$.

Eg:- $\langle a_n \rangle = \langle n \rangle$ diverge to $+\infty$

i.e $\lim_{n\to\infty} a_n = +\infty$

Diverge to $-\infty$:- A seqⁿ $\langle a_n \rangle$ is said to be diverge to $-\infty$ if given $k \in \mathbb{R}$ However large, $\exists$ $m \in \mathbb{N}$ (depending on k) s.t $a_n < -k$ $\forall$ $n \geq m$, and we write $\lim\limits_{n\to\infty} a_n = -\infty$.

Eg:- $\langle -n \rangle$ diverge to $-\infty$ i.e $\lim\limits_{n\to\infty} a_n = -\infty$.

Eg:- $a_n = \langle n^{n^2} \rangle$ diverge to $+\infty$

Eg. $a_n = \langle n \sin n \rangle$ diverge to $+\infty$.

Ultimate sequence :- A seqⁿ is said to be Ultimate if $\exists$ $m \in \mathbb{Z}^+$ s.t $a_n \leq a_{n+1}$ $\forall$ $n \geq m$.

Ex:- $\{1, -\frac{1}{2}, 0, 2, -1, 5, 6, 7, 8, \ldots\}$

Oscillatory sequence :- If a seqⁿ $\langle a_n \rangle$ Neither cgs to a finite Number Nor diverge to $+\infty$ or $-\infty$, then seqn called an oscillatory sequence. It is of two types.

Oscillate seqn of finite oscillation :- A seqn $\langle a_n \rangle$ of Real Number is Bdd but Not cgt to Real Number

OR

It has atleast two limit point is called oscillate seqn of finite oscillation.

Eg:- $\langle a_n \rangle = \langle (-1)^n \rangle$ is bdd but

$\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} (-1)^{2n} = 1$ and

$\lim\limits_{n\to\infty} a_{2n+1} = \lim\limits_{n\to\infty} (-1)^{2n+1} = -1$

Thus $\lim_{n\to\infty} a_n$ doesn't exist ⇒ seq^n doesn't cgt.

i.e $\langle(-1)^n\rangle$ is oscillate finitely.

ii) Oscillate seq^n of Infinite oscillation :-

A unbdd seq^n is called oscillate seq^n of Infinite oscillation iff Neither converge to $+\infty$ Nor converge to $-\infty$.

OR Infinite limit point.

Ex:- $\langle a_n\rangle = \langle(-1)^n \cdot n\rangle$

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n}\cdot n = +\infty.$$

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1}\cdot n = -\infty.$$

i.e Neither converge to $+\infty$ Nor converge to $-\infty$

i.e Oscillate Infinitely.

Note :- when we say $\lim_{n\to\infty} a_n = l$ it means

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = l.$$

Similarily $\lim_{n\to\infty} a_n = +\infty$ or $-\infty$ it means

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = +\infty \text{ or } -\infty.$$

Eg:- $\langle (n)^{(-1)^n}\rangle$

seq^n is unbdd.

$(n)^{(-1)^n} = \{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \text{------}\} = \begin{cases} n ; & \text{if } n \text{ is even} \\ \frac{1}{2n+1} ; & \text{if } n \text{ is odd} \end{cases}$

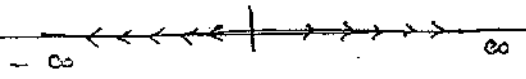
Not diverge to $+\infty$.

i'e Infinite oscilation.

Ex:- $\langle (-n)^{(-1)^n} \rangle = \{-1, -2, \frac{-1}{3}, -4, \frac{-1}{5}, \text{----}\}$

Infinite oscilation (Not diverge to $-\infty$).

Eg:- $\{-1, 2, -3, 4, -5, 6, \text{-- --}\}$



infinite oscillation.

Eg:- $\{m + \frac{1}{n} ;\ m, n \in \mathbb{N}\}$

Infinite oscillation. such it has Infinite limit point.

Null sequence:- A seq^n $\langle a_n \rangle$ is said to be Null seq^n. if it converges to zero. i'e $\lim_{n\to\infty} a_n = 0$.

Eg:- 1> $\left\langle \frac{1}{n} \right\rangle$ 2> $\left\langle \frac{1}{n^2} \right\rangle$, 3> $\left\langle \frac{1}{2^n} \right\rangle$

4> $\left\langle \frac{(-1)^{n-1}}{n} \right\rangle$

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Result:- if $\langle a_n \rangle$ ~~cges~~ converges to 'l' , then the seq^n $\langle |a_n| \rangle$ converges to $|l|$.

i.e if $\langle a_n \rangle \longrightarrow l$ then $\langle |a_n| \rangle \longrightarrow |l|$ as $n \to \infty$.

Note:- converse of above Result is Not true.

Eg:- if $\langle a_n \rangle = \langle (-1)^n \rangle$

Hence $\langle |a_n| \rangle = \langle 1 \rangle$ (constant Seq^n).

$\langle |a_n| \rangle \longrightarrow 1$ as $n \to \infty$ but $\langle a_n \rangle = \langle (-1)^n \rangle$ doesn't converge.

Th^m:- if seq^n $\langle a_n \rangle$ Diverges to $+\infty$ then $\langle a_n \rangle$ is bounded below but unbdd above.

Th^m:- if seq^n $\langle a_n \rangle$ Diverges to $-\infty$ then $\langle a_n \rangle$ is bdd above but Not bdd below.

Notes:- converse of above these two Th^m are Not true.

Eg:- (i) $$a_n = \begin{cases} n & \text{if } n \text{ is even} \\ 1/n & \text{if } n \text{ is odd} \end{cases}$$

Hence $\langle a_n \rangle$ is bdd below and unbdd above but $\langle a_n \rangle$ doesn't diverge to ∞.

(ii) $$a_n = \begin{cases} -n & \text{if } n \text{ is even} \\ -1/n & \text{if } n \text{ is odd} \end{cases}$$

Hence a_n is bdd above by zero. But

unbdd below but $\langle a_n \rangle$ doesn't diverge to $-\infty$.

Note:- Constant seq^n is Monotonic Increasing and Monotonic Decreasing.

Eg:- $\{0, 1, 2, 3, 3, 3, --\}$
Monotonic Increasing seq^n But Not S.M.I.S.

Eg:- $\langle n \rangle$ - S.M.I or Increasing.
Eg:- $\langle -n \rangle$ - S.M.D or Decreasing.

Eg:- $\{0, -1, -2, -3, -3, -3, ---\}$
Monotonic Decreasing seq^n But Not S.M.D.S.

Def^n: Ultimate Monotonically Increasing:- A seq^n $\langle a_n \rangle$ of Real No. is called U.M.I if $\exists$ a +ve integer k s.t $a_{n+1} \geq a_n$ $\forall$ $n \geq k$.

$\{0, 1, -1, -\frac{1}{2}, 3, 3, 3, ------\}$

Th^m:- Every Monotonically Increasing seq^n which is bdd above converges to it's least upper Bound.

Th^m:- Every Monotonically decreasing seq^n which is bdd below converges to it's greatest lower bound.

Th^m:- The Necessary and Sufficient condition for the convergence of a Monotonic seq^n is that is Bounded.

(i) $\langle a_n \rangle$ is monotonically I & bdd $\Rightarrow$ converges to its l.u.b

ii) an is monotonically ↓ and bdd ⇒ converges to its g.l.b.

Th^m:- Every monotonically increasing sequence which is Not bdd above diverge to ∞.

Th^m: Every monotonically decreasing seq^n which is Not bdd below diverge to $-\infty$.

Theorem: Every Monotonic seq^n either converges or diverges Never oscillates.

Case → I $\langle a_n \rangle$ is Monotonically ↑.

if bdd above → converges to its L.U.B.

if Not bdd above → diverge to $+\infty$

Case – II $\langle a_n \rangle$ is Monotonically ↓

if bdd below → Converges to its g.l.B.

if unbdd below → Diverge to $-\infty$

Note →1 A seq^n $\langle a_n \rangle$ is Null seq^n iff the seq^n $\langle |a_n| \rangle$ is Null sequence.

i.e $\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$

Note →2 A seq^n $\langle a_n \rangle$ converges to l iff $\langle a_n - l \rangle$ converges to zero i.e $\langle a_n - l \rangle$ is Null seq^n

i.e $\langle a_n \rangle$ converges to $l \iff \langle a_n - l \rangle \to 0$

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Note→3 if $|a_n| \le |b_n|$ $\forall$ n and $\langle b_n \rangle$ is Null seq^n then $\langle a_n \rangle$ is also Null seq^n.

Note→4 if seq^n $\langle a_n \rangle$ converges to zero and $\langle b_n \rangle$ is bdd seq^n then $\langle a_n \cdot b_n \rangle$ is Null Sequence.

Note→5 if a seq^n $\langle a_n \rangle$ oscillates finitely and $\lim_{n\to\infty} b_n = 0$ then $\lim_{n\to\infty} (a_n \cdot b_n) = 0$

oscillates finitely ⇒ bdd seq^n.

Note→6 if $\langle a_n \rangle$ converges to zero and k be any constant then seq^n $\langle k \cdot a_n \rangle$ is also converges to zero.

Note→7 if a sequence is convergent and converges to 'l' then its every subsequence will be convergent and converges to l.

Note→8 If two complementry subsequence of a seq^n $\langle a_n \rangle$ converges to same limit then seq^n $\langle a_n \rangle$ convergent and converges to same limit.

i.e if $\langle a_{2n} \rangle$ & $\langle a_{2n+1} \rangle$ be two complementry subseq^n of $\langle a_n \rangle$.

and $\lim_{n\to\infty} a_{2n} = l$ and $\lim_{n\to\infty} a_{2n+1} = l$ then

$$\boxed{\lim_{n\to\infty} a_n = l.}$$

Note→9 Every limit point of subseq^n is also limit point of seq^n. Not conversely.

Eg = $\langle a_n \rangle = (-1)^n$ wh. with two subseq^n

$\langle a_{2n} \rangle = \langle 1 \rangle$ &

$\langle a_{2n+1} \rangle = \langle -1 \rangle$

$\langle a_{2n} \rangle$ and $\langle a_{2n+1} \rangle$ has limit point 1 and -1 Respectively then 1 & -1 also limit points of $\langle a_n \rangle$

But 1 & -1 which are two the limit points of $\langle a_n \rangle$, are Not limit points of its subseq^n $\langle a_{2n} \rangle$.

10> If a subseq^n has a limit then that limit will be limit point of seq^n.

11> A seq^n has no limit point iff seq^n has No convergent Subsequence.

12> If a sequence has limit point then it has a subseq^n which converges to that limit point

13> A seq^n has limit point iff it has a cgt subsequence.

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Algorithm for showing $\lim_{n\to\infty} a_n = l$.

Step 1 Simplify $|a_n - l|$ to get

$|a_n - l| \leq \phi(n)$ [function of n]

Step-2 Solve $\phi(n)$ to get

$n > \psi(\epsilon)$ (Function of ϵ)

Step→3 Validation step.

if $\psi(\epsilon)$ is defined $\forall\ \epsilon > 0$ then set $m = [\psi(\epsilon)]$.

Step 4 claim $n \geq m \Rightarrow |a_n - l| < \epsilon$.

Eg:- $\langle a_n \rangle = \sqrt{n+1} - \sqrt{n}$. Show $\lim_{n \to \infty} \langle a_n \rangle = 0$

Soln:- $a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{\sqrt{n+1} - \sqrt{n} \times \sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$

$$a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Step→1 $|a_n - 0| = \left| \dfrac{1}{\sqrt{n+1} + \sqrt{n}} \right| < \dfrac{1}{\sqrt{n} + \sqrt{n}} = \dfrac{1}{2\sqrt{n}} < \epsilon$

Hence $\phi(n) = \dfrac{1}{2\sqrt{n}}$

Step→2 $\dfrac{1}{2\sqrt{n}} < \epsilon \Rightarrow \dfrac{1}{4n} < \epsilon^2 \Rightarrow n > \dfrac{1}{4\epsilon^2}$

Step→3 Here $\psi(\epsilon) = \dfrac{1}{4\epsilon^2}$ and valid $\forall\ \epsilon > 0$

then $m = \left[\dfrac{1}{4\epsilon^2} \right]$

Step→4 $\forall\ n \geq m$? $|a_n - 0| < \epsilon$

Hence 0 is limit of seqn $\langle a_n \rangle$.

Qus Show that seq^n $\langle \log \frac{1}{n} \rangle$ Diverge to $-\infty$.

Sol^n:- If any seq^n diverges to $-\infty$ then for given $k \in \mathbb{R}^+$ however large $\exists$ $m \in \mathbb{N}$ s.t. $a_n < -k$ $\forall$ $n \geq m$.

So on this behalf we are going to find such m for which $a_n < -k$ $\forall$ $n \geq m$.

So let $k > 0$ be given & $a_n = \log \frac{1}{n}$

$$a_n < -k$$
$$\log \frac{1}{n} < -k$$

$$-\log n < -k \quad \{\log n^{-1} = -\log n\}$$
$$\log n > k$$
$$n > e^k$$

If m is +ve integer $> e^k$ then $a_n < -k$ $\forall$ $n \geq m$.

$$\Rightarrow \lim_{n \to \infty} a_n = -\infty.$$

Qus By defination show that seq^n $\langle \frac{1}{3^n} \rangle$ converges to zero.

Sol^n:- $|a_n - 0| = \left| \frac{1}{3^n} - 0 \right| = \left| \frac{1}{3^n} \right| = \frac{1}{3^n} < \epsilon$

So $\phi(n) = \frac{1}{3^n}$. Now we get $\psi(\epsilon)$

$$\frac{1}{3^n} < \epsilon \rightarrow 3^n > \frac{1}{\epsilon}$$

$$\Rightarrow n \log 3 > \log\left(\frac{1}{\epsilon}\right)$$

$\left\{ x_1 < x_2 \Rightarrow \log x_1 < \log x_2 \right.$ since log fun ↑. $\left.\right\}$

$$\Rightarrow n > \frac{\log(1/\epsilon)}{\log 3}$$

Hence if $n \geq m$ $\left[m = \frac{\log(1/\epsilon)}{\log 3} \right]$ then $|a_n - 0| < \epsilon$ $\forall\ n \geq m$

$$\Rightarrow \lim_{n\to\infty} a_n = 0.$$

Qu if $x_n = 1 + \frac{(-1)^n}{2n}$. find least +ve integer s.t

$|x_n - 1| < \frac{1}{10^3}$ $\forall\ n > m$.

Soln. $|x_n - 1| < \frac{1}{10^3}$

$$\Rightarrow \left| \frac{(-1)^n}{2n} \right| < \frac{1}{10^3}$$

$$\Rightarrow \left| \frac{(-1)^n}{2n} \right| = \frac{1}{2n} < \frac{1}{10^3}$$

$$2n > 10^3$$

$$n > 500$$

By taking $m = 500$, we have $|x_n - 1| < \frac{1}{10^3}$ $\forall\ n > m$

Qu if $a_n = 2 + \frac{(-1)^n}{n^2}$, find the least +ve integer

m s.t $|a_n - 2| < \frac{1}{10^4}$ $\forall\ n > m$.

Soln:- $|a_n - 2| < \frac{1}{10^4}$

$\Rightarrow \left| 2 + \frac{(-1)^n}{n^2} - 2 \right| = \left| \frac{(-1)^n}{n^2} \right| < \frac{1}{10^4}$

$\frac{1}{n^2} < \frac{1}{10^4}$

$n^2 > 10^4$

$n^2 > 10000$

$n > \sqrt{10000} = 5000$

i.e by taking $m = 5000$, we have $|a_n - 2| < \frac{1}{10^4}$ $\forall\ n > m$.

Q3 if $a_n = c$ for all $n \in \mathbb{N}$ is a constant seqn then $\lim\limits_{n\to\infty} a_n = c$.

Soln:- $a_n = c \ \forall\ n$

$\Rightarrow |a_n - c| = 0 \ \forall\ n$.

for each $\epsilon > 0$ and for each $m \in \mathbb{N}$, we have.

$|a_n - c| = 0 < \epsilon \ \forall\ n \geq m \Rightarrow \lim\limits_{n\to\infty} a_n = c$.

Q4 Show that the seqn $\langle (-1)^n \rangle$ Doesn't converge

$\langle a_n \rangle = \langle -1, 1, -1, 1, - - - - \rangle$

seqn is bdd but Not unique limit point

$\Rightarrow$ Not cgt.

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Thm:- if $\lim_{n\to\infty} a_n = a$ and $a \neq 0$ then there exist a positive Number K and positive integer m s.t $|a_n| \geq K \ \forall \ n \geq m$.

Thm:- if $a_n \to a$ and $b_n \to b$ and $a_n \leq b_n \ \forall \ n$ then $a \leq b$.

Thm:- Sandwich Theorem or Squeeze Principle:-

if $\langle a_n \rangle$, $\langle b_n \rangle$ & $\langle c_n \rangle$ are three sequences s.t

i) $a_n \leq b_n \leq c_n \ \forall \ n$.

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$ then

$\lim_{n\to\infty} b_n = l$.

Algebra of limit of seq^{n}s:-

Thm. If $\langle a_n \rangle \longrightarrow a$ and $\langle b_n \rangle \longrightarrow b$ then

1) $\langle a_n \pm b_n \rangle \longrightarrow a \pm b$

2) $\langle a_n \cdot b_n \rangle \longrightarrow a \cdot b$

3) $\left\langle \frac{a_n}{b_n} \right\rangle \longrightarrow \frac{a}{b}$ $(b_n \neq 0)$

4) $\langle |a_n| \rangle \longrightarrow |a|$

Note:- if any one of seq^n $\langle a_n \rangle$ and $\langle b_n \rangle$ Not converges then we can't say about their sum, difference, Product and division.

Eg:- 1) $a_n \to \infty$, $b_n \to -\infty$ but $(a_n + b_n) \longrightarrow$ cgs.

Eg:- $a_n = n$, $b_n = -n$

then $\langle a_n + b_n \rangle = \langle 0 \rangle$ cgs.

2) $a_n \to \infty$, $b_n \to -\infty$ but $(a_n+b_n) \to$ diverge

E.g:- $a_n = n$, $b_n = -2n$ then

$(a_n+b_n) = (-n)$ divergent.

3) $a_n \to \infty$, $b_n \to -\infty$ but $(a_n+b_n) \to$ oscillate

E.g:- $\{1, -3, 5, -7, \dots\} = a_n$

& $b_n = \{-2, 4, -6, 8, \dots\}$

then $a_n + b_n = \{-1, 1, -1, 1, \dots\}$

$= (-1)^n \to$ oscillate.

4) $a_n \to \infty$, $b_n \to$ converge but $(a_n \cdot b_n) \to$ divergent

E.g:- $a_n = n^2$, $b_n = \frac{1}{n}$ then $(a_n \cdot b_n) = n \to$ divergent

5) $a_n \to \infty$, $b_n \to$ cgt but $(a_n \cdot b_n) \to$ cgt.

e.g. $a_n = n$, $b_n = \frac{1}{n}$ then $(a_n b_n) \to$ cgt.

6) $a_n \to \infty$, $b_n \to$ cgt but $(a_n \cdot b_n) \to$ oscillate

E.g:- $a_n = n$, $b_n = \frac{(-1)^n}{n}$ then $(a_n \cdot b_n) = (-1)^n \to$ oscillate

IMP:-
Thm. If $\langle a_n \rangle$ is a seqn of $a_n \neq 0$ for any n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ where $|l| < 1$ then $\lim_{n\to\infty} a_n = 0$

And if $|l| > 1$, then $\lim_{n\to\infty} a_n = \infty$

Thm:- The set of limit points of a seqn is closed set.

Thm:- The ~~seq~~ set of limit point of a Bdd seqn is a compact set.

Limit Inferior And Limit Superior of a Sequence.

Case-I Sequence $\langle a_n \rangle$ is Bounded Sequence.

As we know if $\langle a_n \rangle$ is bdd seqn, then $\langle a_n \rangle$ has least and greatest limit points so,

1> The least limit point of $\langle a_n \rangle$ is called limit inferior (or inferior limit or lower limit) of $\langle a_n \rangle$ And denoted by $\lim_{n\to\infty} \text{Inf.} \, a_n$ or simply by $\underline{\lim}\, a_n$

2> The greatest limit point of $\langle a_n \rangle$ is called limit superior (or superior limit or upper limit) of $\langle a_n \rangle$ And denoted by $\lim_{n\to\infty} \text{Sup}\, a_n$ or simply by $\overline{\lim}\, a_n$

Case-II Sequence $\langle a_n \rangle$ is Unbounded Sequence.

1> if $\langle a_n \rangle$ is Not bdd by above then

$\overline{\lim}\, a_n = \lim_{n\to\infty} \sup a_n = +\infty$.

2> if $\langle a_n \rangle$ is Not bdd by below then.

$\underline{\lim}\, a_n = \lim_{n\to\infty} \inf a_n = -\infty$.

Case-III $\langle a_n \rangle$ be seqn of Real Number. May or May Not be Bounded [General Def^n]

Limit Inferior of seq^n $<a_n>$:- Let $<a_n>$ be seq^n of Real Number is limit Inferior of seq^n $<a_n>$ is $\sup\limits_n \inf\{a_n, a_{n+1}, ---\}$

It is denoted by $\underline{\lim}\, a_n$ or $\underline{\lim}_{n\to\infty} a_n$

i.e $\boxed{\liminf\limits_{n\to\infty} a_n = \sup\limits_n \inf\{a_n, a_{n+1}, ---\}}$

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$\underline{\lim}\, a_n = \sup A_n$

where $A_n = \inf\{a_n, a_{n+1}, ---\}$

$n=1, \quad A_1 = \inf\{a_1, a_2, ---\} = \alpha_1$

$n=2, \quad A_2 = \inf\{a_2, a_3, ---\} = \alpha_2$

$\vdots$

$n=n, \quad A_n = \inf\{a_n, a_{n+1}, ---\} = \alpha_n$

$\vdots$

$\underline{\lim}\, a_n = \sup\limits_n A_n$

$= \sup\{A_1, A_2, ---\}$

$= \sup\{\alpha_1, \alpha_2, ---\}$

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Limit Superior of seq^n $<a_n>$:- A seq^n of $<a_n>$ of Real Number is limit Superior of a_n is $\inf \sup\{a_n, a_{n+1}, ---\}$

It is denoted by $\overline{\lim}\, a_n$ or $\overline{\lim}_{n\to\infty} a_n$

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i'e $\boxed{\overline{\lim_{n\to\infty}}\, a_n = \inf \sup \{a_n, a_{n+1}, \ldots\}}$

$\overline{\lim}\, a_n = \inf\limits_{n} B_n$

where

$B_n = \sup\{a_n, a_{n+1}, \ldots\}$

$n=1$, $B_1 = \sup\{a_1, a_2, \ldots\} = \beta_1$

$B_2 = \sup\{a_2, a_3, \ldots\} = \beta_2$

$B_3 = \sup\{a_3, a_4, \ldots\} = \beta_3$

$\vdots$

$B_n = \sup\{a_n, a_{n+1}, \ldots\} = \beta_n$

$\vdots$

$\overline{\lim}\, a_n = \inf\{B_1, B_2, \ldots\}$

$= \inf\{\beta_1, \beta_2, \ldots\}$

i'e $\overline{\lim}\, a_n = \inf\{\beta_1, \beta_2, \ldots\}$

Note:- In limsuperior and liminferior we taking about in extended Real Number.

Note:- Four oscillate finitely :- $\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$ [both finite].

For oscillate infinitely :- At least one of $\overline{\lim}\, a_n$ or $\underline{\lim}\, a_n$ is not finite.

Exs:- $\langle a_n \rangle = \langle \frac{1}{n} \rangle$

As we know that $\langle \frac{1}{n} \rangle$ is bdd seqn and 0 is only limit point. So by.

Case I :- '0' is upper and lower limit. So $\overline{\lim}\, a_n = \underline{\lim}\, a_n = 0$.

D.4

By General defination;

$\underline{\lim} \langle a_n \rangle = \sup\{\inf\{a_n, a_{n+1}, \ldots\}\}$

$A_n = \inf\{n, n+1, \ldots\}$

$n=1, \; A_1 = \inf\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} = 0$

$A_2 = \inf\{\frac{1}{2}, \frac{1}{3}, \ldots\} = 0$

|

|

$A_n = \inf\{\frac{1}{n}, \frac{1}{n+1}, \ldots\}$

|

$\sup\{A_1, A_2, A_3, \ldots\}$

$\sup\{0, 0, 0, \ldots\}$

$= 0$

i.e $\underline{\lim} \langle a_n \rangle = 0$.

Similarily.

$\overline{\lim} \langle a_n \rangle = \inf\{\sup\{n, n+1, \ldots\}\}$

$B_n = \sup\{x_n, x_{n+1}, - - - -\}$.

$n=1$, $B_1 = \sup\{1, \frac{1}{2}, \frac{1}{3}, - - - -\} = 1$

$B_2 = \sup\{\frac{1}{2}, \frac{1}{3}, - - - -\} = \frac{1}{2}$

$B_3 = \sup\{\frac{1}{3}, \frac{1}{4}, - - - -\} = \frac{1}{3}$

$\vdots$

$B_n = \sup\{\frac{1}{n}, \frac{1}{n+1}, - - -\} = \frac{1}{n}$

$\vdots$

$= \inf\{B_1, B_2, - - - - -\}$

$= \inf\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, - - - - -\}$

$= 0$

i.e. $\overline{\lim}\langle a_n\rangle = 0$

i.e. $\underline{\lim}\langle a_n\rangle = \overline{\lim}\langle a_n\rangle = 0$.

Note: If you are able to find out limit points of a bdd seqn then apply case-I; otherwise you should apply always General defination. And case II is Particular case of General defination.

2) $\langle a_n\rangle = 1+(-1)^n$

This seq^n has only two limits points 0 & 2

So $\underline{\lim}\langle a_n\rangle = 0$ and $\overline{\lim}\langle a_n\rangle = 2$.

3) $\langle a_n\rangle = \langle n\rangle$

This seq^n is unbdd above but bdd below

So $\overline{\lim}\, a_n = +\infty$

and $\underline{\lim}\, a_n = \sup\{\inf\{n, n+1, ---\}\}$

$B_n = \inf\{n, n+1, ---\}$

n=1, $B_1 = \inf\{1, 2, 3, --\} = 1$

$B_2 = \inf\{2, 3, 4, ---\} = 2$

$B_3 = \inf\{3, 4, 5, ---\} = 3$

$\vdots$

$B_n = \inf\{n, n+1, ---\} = n$

$\vdots$

$\underline{\lim}\, a_n = \sup\{1, 2, 3, ----\}$
$= \infty$.

i.e $\overline{\lim}\, a_n = +\infty$ and $\underline{\lim}\, a_n = +\infty$.

i.e $\overline{\lim}\, a_n = \underline{\lim}\, a_n = \infty$

4) $\langle a_n\rangle = \left\langle \frac{(-1)^n}{n^2} \right\rangle$; seq^n $\langle a_n\rangle$ only '0' limit point.

So $\overline{\lim} = \underline{\lim} = 0$.

5) $\langle a_n\rangle = \langle n(1+(-1)^n)\rangle$; seq^n is bdd below and unbdd above.

i.e $\overline{\lim}\langle a_n\rangle = \infty$

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$\underline{\lim} \langle a_n \rangle = \sup\{\inf\{n, n+1, ---\}\}$

$B_1 = \inf\{n, n+1, ---\}$

$B_1 = \inf\{0, 4, ---\} = 0$

$B_2 = \inf\{4, 0, ---\} = 0$

|

|

|

$B_n = \inf\{ n(1+(-1)^n),$ ~~n(1~~ $n+1(1+(-1)^{n+1}, ---\} = 0$

|

|

$\underline{\lim} \langle a_n \rangle = \sup\{0, 0, 0, ---\}$

$= 0$

i.e $\overline{\lim} \langle a_n \rangle = \infty$ and $\underline{\lim} \langle a_n \rangle = 0$

6) $\langle a_n \rangle = \langle \sin \frac{n\pi}{3} \rangle$

$\langle a_n \rangle = \{ \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0 --- \}$

Here $0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$ are Repeating infinite times so these are the only limit points So.

$\overline{\lim}\, a_n = \frac{\sqrt{3}}{2}$ & $\underline{\lim}\, a_n = -\frac{\sqrt{3}}{2}$

7). $\langle a_n \rangle = \langle (-1)^n \cdot n \rangle$

This seq^n is bdd below and unbdd Above.

So $\overline{\lim}\, a_n = +\infty$

and $\underline{\lim}\, a_n = \sup\{\inf\{n, n+1, \ldots\}\}$

$B_n = \inf\{n, n+1, \ldots\}$

$n=1,\quad B_1 = \inf\{-1, 2, -3, 4, \ldots\} = -\infty$

$B_2 = \inf\{2, -3, 4, \ldots\} = -\infty$

$\vdots$

$B_n = \inf\{(-1)^n n, (-1)^{n+1} n+1, \ldots\} = -\infty$

$\underline{\lim}\, a_n = \sup\{-\infty, -\infty, \ldots\} = -\infty.$

i.e $\underline{\lim}\, a_n = -\infty.$

i.e $\overline{\lim}\, a_n = +\infty$ and $\underline{\lim}\, a_n = -\infty.$

8) $a_n = \left\langle \cos \frac{n\pi}{2} \right\rangle$

$\langle a_n \rangle = \{0, -1, 0, 1, 0, -1, \ldots\}$

i.e only limit point of seqn $\langle a_n \rangle$ is $0, 1, -1.$

So $\overline{\lim}\langle a_n \rangle = 1,\quad \underline{\lim}\langle a_n \rangle = -1.$

9) $a_n = \left\langle 1 + \frac{1}{n} \right\rangle$

seqn has only one limit point '1'.

$\overline{\lim}\langle a_n \rangle = 1 = \underline{\lim}\langle a_n \rangle$

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10) $\langle a_n \rangle = \langle (-1)^n \rangle$

$= \{-1, 1, -1, 1, \ldots\}$

This seq^n has two limit point -1 & 1

$\overline{\lim} \langle a_n \rangle = 1$ & $\underline{\lim} \langle a_n \rangle = -1$

11) $\langle a_n \rangle = \langle \pi n \rangle$

This seq^n is bdd below and unbdd above.

so $\overline{\lim} \langle a_n \rangle = +\infty$.

and $\underline{\lim} \langle a_n \rangle = \sup\{\inf\{n, n+1, \ldots\}\}$

$B_n = \inf\{n, n+1, \ldots\}$

$n=1$, $B_1 = \inf\{\pi, 2\pi, \ldots\} = \pi$

$B_2 = \inf\{2\pi, 3\pi, \ldots\} = 2\pi$

|

|

$B_n = \inf\{n\pi, (n+1)\pi, \ldots\} = n\pi$

|

|

$\sup\{\pi, 2\pi, 3\pi, \ldots\} = \infty$

i.e $\underline{\lim} \langle a_n \rangle = +\infty$.

i.e $\overline{\lim} \langle a_n \rangle = \underline{\lim} \langle a_n \rangle = +\infty$.

Notes:- Let $\underline{\lim}\, a_n = \alpha$ and $\overline{\lim}\, a_n = \beta$ then $a_n \notin (\alpha, \beta)$ $\forall\, n \in \mathbb{N}$ (possible but Not always). But in every Nbd of $\overline{\lim}$ and $\underline{\lim}$, there are infinite member of sequence.

Result:- If $\langle a_n \rangle$ be any seq^n, then

$$\boxed{\text{Inf } a_n \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq \sup a_n}$$

Result:- if $\langle a_n \rangle$ & $\langle b_n \rangle$ are two bdd seq^n. s.t $a_n \leq b_n$ then.

1) $\overline{\lim} \cdot a_n \leq \overline{\lim}\, b_n$
2) $\underline{\lim}\, a_n \leq \underline{\lim}\, b_n$

Result:- Let $\langle a_n \rangle$ and $\langle b_n \rangle$ are bdd seq^n of +ve Real Number then.

1) $\underline{\lim}\, a_n + \underline{\lim}\, b_n \leq \underline{\lim}\,(a_n+b_n) \leq \underline{\lim}\, a_n + \overline{\lim}\, b_n$

2) $\underline{\lim}\, a_n + \overline{\lim}\, b_n \leq \overline{\lim}\,(a_n+b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$

3) $\underline{\lim}\, a_n \cdot \underline{\lim}\, b_n \leq \underline{\lim}\, a_n \cdot b_n \leq \underline{\lim}\, a_n \cdot \overline{\lim}\, b_n$

4) $\underline{\lim}\, a_n \cdot \overline{\lim}\, b_n \leq \overline{\lim}\,(a_n \cdot b_n) \leq \overline{\lim}\, a_n \cdot \overline{\lim}\, b_n$

5). $\underline{\lim}\, a_n - \overline{\lim}\, b_n \leq \underline{\lim}\,(a_n-b_n) \leq \begin{cases} \underline{\lim}\, a_n - \underline{\lim}\, b_n \\ \text{or } \overline{\lim}\, a_n - \overline{\lim}\, b_n \end{cases}$

$\leq \overline{\lim}\,(a_n-b_n) \leq \overline{\lim}\, a_n - \underline{\lim}\, b_n$

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Results:- if $\langle a_n \rangle$ and $\langle b_n \rangle$ are two bdd seq^n then.

1) $\overline{\lim}\,(a_n+b_n) \leq \overline{\lim} \cdot a_n + \overline{\lim}\, b_n$
2) $\overline{\lim}\,(a_n+b_n) \geq \overline{\lim}\, a_n + \underline{\lim}\, b_n$
3) $\overline{\lim}\,(a_n+b_n) \geq \underline{\lim}\, a_n + \overline{\lim}\, b_n$
4) $\underline{\lim}\,(a_n+b_n) > \underline{\lim}\, a_n + \underline{\lim}\, b_n$

5) $\underline{\lim}\,(a_n+b_n) \leq \overline{\lim}\,a_n + \underline{\lim}\,b_n$

6) $\underline{\lim}\,(a_n+b_n) \leq \underline{\lim}\,a_n + \overline{\lim}\,b_n$

7) $\overline{\lim}\,(a_n\cdot b_n) \leq (\overline{\lim}\,a_n)\cdot(\overline{\lim}\,b_n)$

8) $\overline{\lim}\,(a_n\cdot b_n) \geq (\underline{\lim}\,a_n)\cdot(\overline{\lim}\,b_n)$

9) $\underline{\lim}\,(a_n\cdot b_n) \geq (\underline{\lim}\,a_n)\cdot(\underline{\lim}\,b_n)$

10) $\underline{\lim}\,(a_n\cdot b_n) \leq (\underline{\lim}\,a_n)\cdot(\overline{\lim}\,b_n)$

11) $\underline{\lim}\left(\frac{1}{a_n}\right) = 1/\overline{\lim}\,a_n$ if $\overline{\lim}\,a_n > 0$

12) $\overline{\lim}\left(\frac{1}{a_n}\right) = \frac{1}{\underline{\lim}\,a_n}$ if $\underline{\lim}\,a_n > 0$

13) $\underline{\lim}\left(\frac{a_n}{b_n}\right) \geq \frac{\underline{\lim}\,a_n}{\overline{\lim}\,b_n}$ if $\overline{\lim}\,b_n > 0$

14) $\overline{\lim}\left(\frac{a_n}{b_n}\right) \leq \frac{\underline{\lim}\,a_n}{\underline{\lim}\cdot b_n}$ if $\underline{\lim}\,b_n < 0$

15) $\overline{\lim}\,(-a_n) = -\underline{\lim}\,a_n$

16) $\underline{\lim}\,(-a_n) = -\overline{\lim}\,a_n$

17) $\overline{\lim}\,(\lambda\cdot a_n) = \lambda\cdot\overline{\lim}\,a_n$ if $\lambda > 0$

18) $\underline{\lim}\,(\lambda\cdot a_n) = \lambda\cdot\underline{\lim}\,a_n$ if $\lambda > 0$

19) $\overline{\lim}\,(\lambda\cdot a_n) = \lambda\cdot\underline{\lim}\,a_n$ if $\lambda < 0$

20) $\underline{\lim}\,(\lambda\cdot a_n) = \lambda\cdot\overline{\lim}\,a_n$ if $\lambda < 0$

Result:- Every sequence contains a Monotonic Subsequence

Result :- Every Bounded sequence in $\mathbb{R}$ contains a cgt Subsequence.

Result :- 1) for convergence

$\overline{\lim}\, a_n = \underline{\lim}\, a_n =$ limit value.

2) for Divergence

$\overline{\lim}\, a_n = \underline{\lim}\, a_n = +\infty$ [or $-\infty$].

Note :- $\underline{\lim}\, a_n$, $\overline{\lim}\, a_n$ are unique.

Note :- $\underline{\lim}\, a_n$, $\overline{\lim}\, a_n$ are finite for Bdd seq^n.

Note :- Limit Inferior, limit sup. of oscillating finite and oscillating infinite is smallest limit point, greatest limit point Respectively.

Note :- A seq^n is unbdd above and Bdd below.

$\overline{\lim}\, a_n = +\infty$

$\underline{\lim}\, a_n = +\infty$ if No limit point.

If the limit point exist then smallest limit point is $\underline{\lim}\, a_n$. since $\underline{\lim}\, a_n =$ smallest limit point

Ex $a_n = \langle n^{(-1)^n} \rangle$ unbdd above & bdd below.

$\lim \sup = +\infty$ $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 9 \cdots\}$

$\lim \inf = 0$

Not diverge to $+\infty$.

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Note :- Similarly unbdd below and bdd above then limit sup and limit inferior is $-\infty$.

X Eg:- $\langle (-1)^n n \rangle = \{-1, 2, -3, 4, -5, - - - \}$

$\lim\sup \to \infty$

$\lim\inf \to -\infty$.

Note:- Seq^n is Diverge to $-\infty$ and diverge to $+\infty$ then $\overline{\lim}\, a_n = \infty$ and $\underline{\lim}\, a_n = -\infty$

Eg: $\langle (-1)^n n \rangle = \{-1, 2, -3, 4, -5, - - - \}$

$\overline{\lim}\, a_n = \infty$

$\underline{\lim}\, a_n = -\infty$.

Note: Sequence has infinite limit point

$\overline{\lim}\, a_n$ = largest limit point

$\underline{\lim}\, a_n$ = smallest limit point

Cauchy Sequence :- Let $\langle a_n \rangle$ be seq^n of Real Number is said to be cauchy seq^n if for every $\epsilon > 0$ $\exists$ a +ve integer m s.t

$$|a_{n+p} - a_n| < \epsilon \quad \forall\ n \geq m,\ p \geq 1$$

Note # A sequence is cauchy iff it is cgt in $\mathbb{R}$.

Note :- A sequence is cauchy iff it is cgt in $\mathbb{C}$.

Algebra of limits :-

Let $\lim a_n = l$ and $\lim b_n = m$ Then

1) $\lim(a_n + b_n) = \lim a_n + \lim b_n = l + m$

(ii) $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n = l \cdot m$

(iii) $\lim(c a_n) = c \lim(a_n) = cl$

(iv) $\lim \max\{a_n, b_n\} = \max\{l, m\}$

(v) $\lim \min\{a_n, b_n\} = \min\{l, m\}$

(vi) $\lim|a_n| = |\lim a_n|$

(vii) if $a_n \geq 0 \ \forall\ n$ $\lim \sqrt{a_n} = \sqrt{\lim a_n} = \sqrt{l}$

Some Theorems for finding the limit of sequence.

Cauchy's First Theorem on limits :-

Let $\langle u_n \rangle$ be seq^n of Real Number and $\lim\limits_{n\to\infty} u_n = l$

then $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n} u_r}{n} = l$

i.e $\lim\limits_{n\to\infty} \dfrac{u_1 + u_2 + \cdots + u_n}{n} = l$

Note:- Converse of above th^m is Not True.

E.g:- $\langle a_n \rangle = (-1)^n$

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} = \begin{cases} 0 & ; n \text{ is even} \\ -1/n & ; n \text{ is odd} \end{cases}$$

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{a_1 + a_2 + \cdots + a_n}{n} = 0$ but $\langle a_n \rangle$ is Not cgt.

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ThM IInd :- Let $\langle u_n \rangle$ be seqn of Real No. & $u_n \geq 0$ $\forall\ n$ and $\lim_{n\to\infty} u_n = l$ then $\lim_{n\to\infty}(u_1 u_2 \cdots u_n)^{1/n} = l$.

i.e $\lim_{n\to\infty} u_n = l \Rightarrow \lim_{n\to\infty}(u_1 \cdot u_2 \cdots\cdots u_n) = l$.

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Cauchy's Second Theorem on Limits :-

if $\langle a_n \rangle$ be a seqn of all +ve terms and if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, then $\lim_{n\to\infty} (a_n)^{1/n}$ exist. And

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (a_n)^{1/n}$ i.e

$\lim (a_n)^{1/n} = \lim \frac{a_{n+1}}{a_n}$, Provided $\lim \frac{a_{n+1}}{a_n}$ exists.

Note:- Converse of cauchy's second Theorem on limits is Not always true.

ThM Ceasaro's Theorem:- if $\lim_{n\to\infty} a_n = a$ and

$\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = a \cdot b$

ThM:- Every cauchy sequence is Bdd

Eg:- $\langle a_n \rangle = \langle 2^{-n+(-1)^n} \rangle$

Now $(a_n)^{1/n} = 2^{-1+\frac{(-1)^n}{n}}$

So $\lim_{n\to\infty} (a_n)^{1/n} = 2^{-1} = \frac{1}{2}$

But $\frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)+(-1)^{n+1}}}{2^{-n+(-1)^n}} = 2^{-n-1+n+(-1)^{n+1}-(-1)^n}$

$$= \begin{cases} 2^{-1-1-1} = 1/2 & \text{if } n \text{ is even} \\ 2^{-1+1+1} = 2 & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ doesn't exist

Hence $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ may not exist even $\lim (a_n)^{1/n}$ exist.

Some Example Related to above thm:-

17) $\lim\limits_{n\to\infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right] = 1$

Soln: ~~By inspection~~

Now multiplied and ~~divied~~ divided by n.

$\lim\limits_{n\to\infty} n \left[\frac{1}{n\sqrt{n^2+1}} + \frac{1}{n\sqrt{n^2+2}} + \cdots + \frac{1}{n\sqrt{n^2+n}}\right]$

$\lim\limits_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}} + \frac{n}{\sqrt{n^2+2}} + \cdots + \frac{n}{\sqrt{n^2+n}}\right]$

$a_n = \frac{n}{\sqrt{n^2+n}}$.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n\sqrt{1+\frac{1}{n}}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$.

then By cauchy first theorem

$$\lim_{n\to\infty} \frac{a_1+a_2+ - - - +a_n}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}} + \frac{n}{\sqrt{n^2+2}} + - - - + \frac{n}{\sqrt{n^2+n}}\right] = 1$$

Hence $\lim_{n\to\infty} \frac{n}{n}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + - - - + \frac{1}{\sqrt{n^2+n}}\right] = 1$

$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + - - - + \frac{1}{\sqrt{n^2+n}}\right] = 1$$

2> find $\lim_{n\to\infty} (n)^{1/n}$

solⁿ:- $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{n\left[1+\frac{1}{n}\right]}{n} = 1$

By cauchy 2nd thm

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

$$\Rightarrow \boxed{\lim_{n\to\infty} (n)^{1/n} = 1}$$

Q-3 find $\lim_{n\to\infty} \frac{1}{n}\left[1+2^{1/2}+3^{1/3}+ - - - +n^{1/n}\right]$

Solⁿ:- $a_n = (n)^{1/n}$ and $\lim_{n\to\infty} (n)^{1/n} = 1$.

then By cauchy Ist thm,

$\lim_{n\to\infty} \frac{1}{n}\left[1+2^{1/2}+3^{1/3}+ \cdots + n^{1/n}\right] = 1$

4) Find $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+ \cdots +\frac{1}{\sqrt{n+n}}\right] =$

$\lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{n}}+\frac{n}{\sqrt{n+1}}+ \cdots +\frac{n}{\sqrt{n+n}}\right]$

$a_n = \frac{n}{\sqrt{n+n}} = \frac{n}{n^{1/2}\sqrt{1+1}} = \frac{n^{1/2}}{\sqrt{2}}$

$\lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{2}} = \lim_{n\to\infty}\sqrt{\frac{n}{2}} = \infty$

i.e. $\lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{n}}+\frac{n}{\sqrt{n+1}}+ \cdots +\frac{n}{\sqrt{n+n}}\right] = \infty$

$\lim_{n\to\infty} = \left[\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+ \cdots +\frac{1}{\sqrt{n+n}}\right] = \infty$

5) Find $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+ \cdots +\frac{1}{(n+n)^2}\right]$

$\lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{n^2}+\frac{n}{(n+1)^2}+ \cdots +\frac{n}{(n+n)^2}\right]$

Here $a_n = \frac{n}{(n+n)^2}$

$\lim_{n\to\infty} a_n = \frac{n}{n^2[1+1]^2} = \frac{1}{4n^2} = 0$

$\Rightarrow \lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+ \cdots +\frac{1}{(n+n)^2}\right] = 0$

6> if $x_n = \left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\cdot\left(\frac{4}{3}\right)^3 \quad \left(\frac{n+1}{n}\right)^n\right]^{1/n}$ then

$\lim_{n\to\infty} x_n = e$

Let $a_n = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n$

$$a_{n+1} = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\left(\frac{n+2}{n+1}\right)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+2}{n+1}\right)^{n+1} = \left[1+\frac{1}{n+1}\right]^{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e$$

$$\lim_{n\to\infty} \left[1+\frac{1}{n+1}\right]^{n+1} = e$$

~~Buy~~ By Cauchy 2nd thm.

if ~~a~~ $\lim \frac{a_{n+1}}{a_n}$ exist then $\lim \frac{a_{n+1}}{a_n} = \lim (a_n)^{1/n}$

$$\lim\left[\left(\frac{2}{1}\right)\cdot\left(\frac{3}{1}\right)^2\cdot\left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n} = e$$

7> Prove that $\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n} = e$

$a_n = \frac{n^n}{n!}$, $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{n^{\overset{n}{\cancel{n+1}}}\left(1+\frac{1}{n}\right)^{n+1}}{(n+1)\cancel{n!}} \cdot \frac{\times \cancel{n!}}{\cancel{n^n}}$$

$$= \frac{n\left(1+\frac{1}{n}\right)^{n+1}}{(1+n)}$$

$$= \frac{\cancel{n}\left(1+\frac{1}{n}\right)^{n+1}}{\cancel{n}\left(\frac{1}{n}+1\right)}$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

So

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}(a_n)^{1/n} = e$$

i.e $\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n} = e$

8) Find $\lim_{n\to\infty} \dfrac{1+\frac{1}{2}+\frac{1}{3}+ \cdots +\frac{1}{n}}{n}$

$$\lim_{n\to\infty} \frac{1}{n}\left[1+\frac{1}{2}+ \cdots +\frac{1}{n}\right]$$

$$a_n = \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\left[1+\frac{1}{2}+ \cdots +\frac{1}{n}\right] = 0$$

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9) $(n!)^{1/n}$

$\lim\limits_{n\to\infty} (n!)^{1/n} = ?$

$a_n = n!$ $\quad a_{n+1} = (n+1)!$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!} = \lim\limits_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} = \infty$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} (a_n)^{1/n} = \infty.$

i.e $\lim\limits_{n\to\infty} (a_n)^{1/n} = \lim\limits_{n\to\infty} (n!)^{1/n} = \infty.$

10) $\lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^{1/n}$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{1}{n+1} \times \frac{n}{1} = \lim\limits_{n\to\infty} \frac{n}{n\left[1+\frac{1}{n}\right]} = 1$

i.e $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} =$ ~~$a_n)$~~ $\lim\limits_{n\to\infty} (a_n)^{1/n} = 1$

i.e $\lim\limits_{n\to\infty} (a_n)^{1/n} = \lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 1$

Result: ① Let $\langle u_n \rangle$ be the seq^n and $u_n \le v_n$

if $\lim u_n = \infty$ Then $\lim v_n = \infty$

② If $\lim v_n = -\infty \Rightarrow \lim u_n = -\infty$

$-u_n \ge -v_n$

$-v_n \le -u_n$ (same above)

Date 142

1> Show that $\left\langle \frac{(-1)^n}{n} \right\rangle$ is cauchy seqn.

PF:- Let $\epsilon > 0$ be given and let $n > m$

Now $|a_n - a_m| = \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| \leq \left| \frac{(-1)^n}{n} \right| + \left| \frac{(-1)^m}{m} \right|$

$= \frac{1}{n} + \frac{1}{m}$

$< \frac{1}{m} + \frac{1}{m}$ $(\because n > m)$

$= \frac{2}{m} < \epsilon$ whenever $m > \frac{2}{\epsilon}$

Hence

for given $\epsilon > 0$ $\exists$ $m \left(> \frac{2}{\epsilon} \right)$ s.t $|a_n - a_m| < \epsilon$ $\forall$ $n > m$.

$\Rightarrow$ $\langle a_n \rangle = \left\langle \frac{(-1)^n}{n} \right\rangle$ is cauchy seq^n.

2> Show that $\langle a_n \rangle = \left\langle \frac{1}{2^n} \right\rangle$ is a cauchy seq^n.

PF:- Let $\epsilon > 0$ (given) and let $n > m$

Now $|a_n - a_m| = \left| \frac{1}{2^n} - \frac{1}{2^m} \right|$

$= \frac{1}{2^m} - \frac{1}{2^n}$ $\left[\because 2^n > 2^m \Rightarrow \frac{1}{2^n} < \frac{1}{2^m} \right]$

$< \frac{1}{2^m} < \epsilon$ whenere $2^m > \frac{1}{\epsilon}$

whenever $m > \frac{\log 1/\epsilon}{\log 2}$

Hence for $\epsilon > 0$ $\exists$ $m \left(> \frac{\log 1/\epsilon}{\log 2} \right)$ s.t

$|a_n - a_m| < \epsilon$ $\forall$ $n > m$ $\Rightarrow$ $\langle a_n \rangle$ is cauchy seq^n.

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Q→ $a_n = 1+\frac{1}{3}+\frac{1}{5}+ - - - +\frac{1}{2n-1}$. Then $\langle a_n \rangle$ is Not cauchy seqn.

Pf:- for $n > m$

$$|a_n - a_m| = \left|1+\frac{1}{3}+\frac{1}{5}+ - - - +\frac{1}{2n-1} - \left(1+\frac{1}{3}+ - + \frac{1}{2m-1}\right)\right|$$

$$= \left|\frac{1}{2m+1}+\frac{1}{2m+3}+ - - - +\frac{1}{2n-1}\right|$$

Now let $n = 2m$ so that $n > m$

$$|a_{2m} - a_m| = \left|\frac{1}{2m+1}+\frac{1}{2m+3}+ - - - +\frac{1}{4m-1}\right|$$

$$> \frac{1}{4m}+\frac{1}{4m}+ - - - +\frac{1}{4m} = \frac{m}{4m} = \frac{1}{4}$$

$|a_{2m} - a_m| > \frac{1}{4}$

if we take $\epsilon = 1/4$ & $n = 2m$ ∄ any m s.t. $|a_{2m} - a_m| < \frac{1}{4}$ but if $\langle a_n \rangle$ is cauchy then ∀ $\epsilon > 0$ ∃ $m \in \mathbb{N}$ s.t. $|a_n - a_m| < \epsilon$ ∀ $n > m$

⇒ $\langle a_n \rangle$ is Not cauchy seqn.

Q→ $a_n = \left(1+\frac{1}{2}+\frac{1}{3}+ - - - +\frac{1}{n}\right)$ is Not cauchy seqn.

8) $a_n = \left(1 + \frac{1}{4} + \frac{1}{7} + \cdots + \frac{1}{3n+2}\right)$ is Not cauchy seq^n.

(Do Yourself).

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IIT JAM-2012.

Que $\langle x_n \rangle = \langle \sqrt{1}, -\sqrt{1}, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, \cdots \rangle$

if $y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$ for all $n \in \mathbb{N}$ then $\langle y_n \rangle$ is

a) Monotonic

b) Not bdd

c) Bdd but Not cgt.

~~d)~~ convergent

Here, $y_1 = \frac{x_1}{1} = \sqrt{1}$

$y_2 = \frac{x_1 + x_2}{2} = \frac{0}{2} = 0$

$y_3 = \frac{x_1 + x_2 + x_3}{3} = \frac{\sqrt{1} - \sqrt{1} + \sqrt{2}}{3} = \frac{\sqrt{2}}{3}$

$y_4 = 0$

$y_5 = \frac{\sqrt{3}}{5}$

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$\langle y_n \rangle = \langle \sqrt{1}, 0, \frac{\sqrt{2}}{3}, 0, \frac{\sqrt{3}}{5}, 0, \cdots \rangle$

Not Monotonic

$\langle y_{2k} \rangle = \langle 0, 0, \cdots \rangle \to 0$ as $n \to \infty$

and $\langle y_{2k-1}\rangle = \langle 1, \frac{\sqrt{2}}{3}, \frac{\sqrt{3}}{5}, \ldots \rangle \to 0$ as $n\to\infty$

two complementary subseq^n $\langle y_{2k}\rangle$ & $\langle y_{2k-1}\rangle$ of $\langle y_n\rangle$ are cgt and converge to same point 0.
Here $\langle y_n\rangle$ will also converge and converges to 0
Hence option (d) is correct.

IISc-2008

Qu. $\langle a_n\rangle$ be a sequence of distinct Real Number which has no Convergent Subsequences. Then $\lim_{n\to\infty} |a_n|$ is

a) 0 b) ∞ c) 1 d) doesn't exist.

Soln- As we know that $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} |a_n| = 0$ but

given has No convergent subsequence. But if $\lim_{n\to\infty} |a_n| = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow \langle a_n\rangle$ is cgt and It will have

cgt Subsequence So $\lim_{n\to\infty} |a_n| = 0$ Not possible.

Hence $\langle a_n\rangle$ has No Convergent Subsequence.

$\Rightarrow$ $\langle a_n\rangle$ is Not convergent & $\langle a_n\rangle$ has No limit point
and $\langle a_n\rangle$ is Not bdd.

$\Rightarrow$ $\langle a_n\rangle$ is Not oscillatory

$\Rightarrow$ ~~Limit n→~~ $\lim_{n\to\infty} a_n$ — $+\infty$ or $-\infty$
— $+\infty$ and $-\infty$ (both)

$\Rightarrow \lim_{n\to\infty} |a_n| = +\infty$

IISC-2010
Qey

Let $\langle x_n \rangle$ be an unbdd sequence of Non-zero Real Numbers. then.

a) $\langle x_n \rangle$ must have convergent subsequence.
b) $\langle x_n \rangle$ can't have convergent subsequence.
c) ✓ $\left\langle \frac{1}{x_n} \right\rangle$ must have conv. subsequence.
d) $\left\langle \frac{1}{x_n} \right\rangle$ can't have conv. subseqn.

soln:- $\langle x_n \rangle$ is unbdd seqn
i.e Never convergent Nor finite oscillation
$\langle x_n \rangle$ — diverge to $+\infty$
or diverge to $-\infty$.

Or seqn of $\langle x_n \rangle$ either diverge to $+\infty$ or diverge to $-\infty$.
in all possibility.

$$\left\langle \frac{1}{x_n} \right\rangle \longrightarrow 0$$

i.e $\left\langle \frac{1}{x_n} \right\rangle$ must have cgt subseqn

(a) $\langle x_n \rangle = \langle n \rangle$
$\langle 1, 2, 3, \text{— —} \rangle$
$\langle x_n \rangle$ have No convergent subseqn.

b) $\langle x_n \rangle = \langle 1, 2, 1, 4, 1, 6, 1, 8 \text{ — — —} \rangle$
$\langle x_{n'} \rangle = \langle 1, 1, 1, 1 \text{ — — — —} \rangle$ is cgt subseqn
so option (b) is discard.

IISc-2010

Qu) $<x_n>$ be a sequence of Real Numbers s.t. $\sum |x_n-x|$
$= c$ with c finite then.

a) $<x_n>$ may Not be bdd.
b) $<x_n>$ must conv. to x.
c) $<x_n>$ must conv. to $x+c$
d) $<x_n>$ is bdd but Not N^y cgt.

$$\sum |x_n - x| = c$$

~~$\lim_{n\to\infty} |x_n| \to x$~~

$$\lim_{n\to\infty} |x_n - x| \to 0 \quad \text{as } n\to\infty.$$

$$\boxed{\lim_{n\to\infty} x_n = x.}$$

$<x_n>$ must conv. to x.

$\because$ $\sum a_n$ cgt then $\lim_{n\to\infty} a_n = 0$.

$\sum |x_n - x| = c \Rightarrow \sum |x_n - x|$ is convergent.

i.e $\lim_{n\to\infty} \sum |x_n - x| \to 0$

i.e $x_n \to x$ as $n\to\infty$.

NBHM 2005

Qu) Evaluate $\lim_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2+1}\right)$

Soln:- $$\lim_{n\to\infty} \sqrt{n^2+n} - \sqrt{n^2+1} \times \frac{\sqrt{n^2+n} + \sqrt{n^2+1}}{\sqrt{n^2+n} + \sqrt{n^2+1}}$$

$$\lim_{n\to\infty} \left[\frac{n-1}{\sqrt{n^2+n} + \sqrt{n^2+1}}\right] = \lim_{n\to\infty} \frac{\left[1 - 1/n\right]}{\left[1+\frac{1}{n^2}\right] + \left[1+\frac{1}{n^2}\right]} = \frac{1}{2}$$

$\lim_{n\to\infty} \sqrt{n^2+n} - \sqrt{n^2+1} = \frac{1}{2}$ Ans.

NBHM-2006 Que Let $\langle a_n \rangle$ be a seq^n of Real Number. Let $b_n = a_n + a_{n+1}$ for $n = 1, 2, \ldots$, which of the following is always true.

a) if $\langle b_n \rangle$ cgs then $\langle a_n \rangle$ cgs.
b) if $\langle a_n \rangle$ conv then $\langle b_n \rangle$ Diverges
c✓ if $\langle a_n \rangle$ conv then $\langle b_n \rangle$ conv
d) $\langle a_n \rangle$ is Subseq^n of $\langle b_n \rangle$
e) None of these.

$\langle a_n \rangle = (-1)^n$

$b_n = a_n + a_{n+1}$

$b_1 = a_1 + a_2$

$b_2 = a_2 + a_3$

$\vdots$

$\langle b_n \rangle = \langle 0, 0, 0, \ldots, 0 \rangle \to 0$

But $\langle a_n \rangle$ is Not cgt

So (a) is wrong

b) $a_n = \langle 1, 1, 1, \ldots \rangle \to 1$

$b_n = \langle 2, 2, \ldots, 2, \ldots \rangle \to 2$

So option (b) is wrong

and (c) is correct. and (d) is wrong.

So option (c) is correct.

TIFR-2010

Qus The sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots$ converges to

a) $\frac{1+\sqrt{33}}{2}$ b) $\frac{1+\sqrt{32}}{2}$ (c) $\frac{1+\sqrt{30}}{2}$ (d) $\frac{1+\sqrt{29}}{2}$

Soln

$a_1 = \sqrt{7}$

$a_2 = \sqrt{7+\sqrt{7}}$

$a_2 = \sqrt{7+a_1}$

$a_2 = \sqrt{a_1+7}$

i'e $a_{n+1} = \sqrt{a_n+7}$; given $a_1 = \sqrt{7}$

$a_{n+1}^2 = a_n + 7$

$\lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty}[a_n+7]$ $\left\{ \text{let } \lim_{n\to\infty} a_n = l \right]$

$l^2 = l+7$

$l^2 - l - 7 = 0$

$l = \frac{1+\sqrt{1+28}}{2} = \frac{1+\sqrt{29}}{2}$

So option (d) is correct.

NET J-2010

Qus $\sup_n \{\sqrt{n+1}-\sqrt{n}\}$ and $\lim\sup_n \{(\sqrt{n+1}-\sqrt{n}\}$ are

a) $\sqrt{2}+1$ and 0 b) $\frac{1}{\sqrt{2}+1}$ and 0

c) both equal to 0 d) both equal to $\frac{1}{\sqrt{2}+1}$

Soln :- $a_n = \sqrt{n+1}-\sqrt{n}$

$\frac{\sqrt{n+1}-\sqrt{n} \times \sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$ as $n\to\infty$

then. $\overline{\lim}\, a_n = 0$ option(d) is wrong.

$\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Hence $\frac{1}{\sqrt{n+1}+\sqrt{n}}$ is ↓ so it will Max. at $n=1$.

Hence $\sup\{\sqrt{n+1}-\sqrt{n}\} = \frac{1}{\sqrt{2}+1}$ ((b) is correct)

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NET J-2012

Qus $a_n = \sin\frac{\pi}{n}$, $n \in \mathbb{N}$ then sup is

a) 0 and it is attained

b) 0 & Not attained

[✓ c)] 1 and attained

d) 1 & Not attained.

$\langle a_n \rangle = \{0, 1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}, \ldots\} \to 0$ as $n \to \infty$

Here $\sup a_n = 1$ by taking $n=2$.

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NET 09.

Qus $\lim \sup \left\{ \frac{(-1)^n}{2^n}, n \in \mathbb{N} \right\}$ is

a) 0 b) $\frac{1}{4}$ c) 1 d) -1

$\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$

So $\overline{\lim}\, a_n = 0$

NET-D-2009.

Que $a_n = n^2 + \frac{(-1)^n}{n}$, $n \in \mathbb{N}$, then $\langle a_n \rangle$

a) Is convergent. But Not cauchy.

b) Is cauchy But Not convergent

c) Is Not cauchy. But Bounded.

d) Doesn't have a convergent Subsequence

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Soln: $\langle a_{2n} \rangle \to \infty$ as $n \to \infty$.

$\langle a_{2n+1} \rangle \to \infty$ as $n \to \infty$

Hence $\lim_{n\to\infty} \langle a_n \rangle \Rightarrow \infty$

$\langle a_n \rangle$ is Not cgt. [(a) discard].

$\langle a_n \rangle$ is Not cauchy [(b) is discard].

$\because$ $\langle a_n \rangle \to \infty$ as $n \to \infty$ $\Rightarrow \langle a_n \rangle$ is Not bdd above.

Option (c) is discard]

$\langle a_n \rangle$ is divergent then $\langle a_n \rangle$ has No limit point

$\Rightarrow \langle a_n \rangle$ has Not cgt a conv. subseq^n.

Hence (d) is correct.

NET-D-2009

Que:- $a_n = (-1)^n \sin\frac{n\pi}{2}$, $n \in \mathbb{N}$ then.

a) $\underline{\lim}\, a_n = -1$ & $\overline{\lim}\, a_n = 1$

b) $\underline{\lim}\, a_n = 0$ & $\overline{\lim}\, a_n = 1$

c) $\underline{\lim}\, a_n = -1$ & $\overline{\lim}\, a_n = 0$

d) $\underline{\lim}\, a_n = 1$ & $\overline{\lim}\, a_n = -1$

$\langle a_n \rangle = \{-1, 0, 1, 0, -1, 0, 1, - -- \}$.

$\overline{\lim}\, a_n = 1$ and $\underline{\lim}\, a_n = -1$

Questions -

NET→2012

Qus→1 The seqn $\langle a_n \rangle = \langle \sin \frac{n\pi}{2} \rangle$ is limtsup

1) 0 and it is attains
2) 0 and it is Not attains
3) 1 and it is attains
4) 1 and it is Not attain.

$\langle a_n \rangle = \{1, 0, -1, 0, 1, \text{------}\}$

Limit points of $\langle a_n \rangle$ is 1, 0, -1

$\overline{\lim} \langle a_n \rangle = 1$ and it is attains

so option (3) is correct.

NET-2012.

Qus 2 $\limsup \left\{ \frac{(-1)^n}{2^n} ; n \in \mathbb{N} \right\}$

1) 0
2) 1/4
3) 1
4)

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soln.- $\lim\limits_{n\to\infty} \frac{(-1)^n}{2^n} = 0$

$\overline{\lim} \langle a_n \rangle = 0$

NET 2011-J

Qus→3 Which of the following is / are correct-

a) $n \log\left(1 + \frac{1}{n+1}\right) \to 1$ as $n \to \infty$

b) $(n+1)\log\left(1+\frac{1}{n}\right) \to 1$ as $n\to\infty$

c) $n^2\log\left(1+\frac{1}{n}\right) \to 1$ as $n\to\infty$

d) $n\log\left(1+\frac{1}{n^2}\right) \to 1$ as $n\to\infty$

Sol^n:- (a), (b), ~~(d)~~.

J-2012
NET

Q→4 Which of the following is / are correct.

a) $\left(1+\frac{1}{n}\right)^{n+1} \to e$ as $n\to\infty$

b) $\left(1+\frac{1}{n+1}\right)^{n} \to e$ as $n\to\infty$

c) $\left(1+\frac{1}{n}\right)^{n^2} \to e$ as $n\to\infty$

d) $\left(1+\frac{1}{n^2}\right)^{n} \to e$ as $n\to\infty$

Sol^n:- (a), (b)

NET-2011 J

Q.45 If $\{x_n\}$ and $\{y_n\}$ are sequence of real Numbers, which of following is / are true?

a) $\limsup(x_n+y_n) \le \limsup x_n + \limsup y_n$

b) $\limsup(x_n+y_n) \ge \limsup x_n + \limsup y_n$

c) $\liminf (x_n + y_n) \leq \lim(\inf) x_n + \liminf y_n$

d) $\liminf (x_n + y_n) \geq \liminf x_n + \liminf y_n$.

Ans (a), (d). (Result).

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Qus 6 let $a_n = \begin{cases} 1 + \frac{1}{n} ; & n \text{ is even} \\ 1 - \frac{1}{n} ; & n \text{ is odd} \end{cases}$ then

a) $\overline{\lim}\, a_n = -1$; $\underline{\lim}\, a_n = 1$

b) $\overline{\lim}\, a_n = -1$; $\underline{\lim}\, a_n = -1$

c) $\overline{\lim}\, a_n = 1$; $\underline{\lim}\, a_n = 1$

d) $\overline{\lim}\, a_n = 1$; $\underline{\lim}\, a_n = -1$

$\lim_{n \to \infty} 1 + \frac{1}{n} = 1$ n is even

$\lim_{n \to \infty} 1 - \frac{1}{n} = 1$ n is odd.

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Qus 7 let $\lim_{n \to \infty} a_n = l$, Now choose $k \in \mathbb{N}$ set

$A_k = \left\{ n \in \mathbb{N} ; |a_n - l| < \frac{1}{k} \right\}$

a) A_k is empty set

b) A_k is all of $\mathbb{N}$ but finitely many

? c) A_k many finite for some k.

d) None of these.

Que 8 The largest term in the sequence. $x_n = \frac{1000^n}{n!}$; $n = 1, 2, \ldots$ is

a) x_{999} b) x_{1001} c) x_1 d) doesn't exist

$x_n = \downarrow$.

$x_1 = \frac{1000}{1!}$ is largest term in the sequence.

Que 9 The sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$

a) divergent sequence

b) Convergent and it's limit $\le \sqrt{2}$

c) conv. and it's limit $\ge \frac{3}{2}$

d) conv. and it's limit > 2

$$a_1 = \sqrt{2}$$
$$a_2 = \sqrt{2\sqrt{2}} = \sqrt{2a_1}$$
$$a_{n+1} = \sqrt{2a_n}$$
$$a_{n+1}^2 = 2a_n \quad \left\{ \text{let } \lim_{n\to\infty} a_n = l. \right.$$

$$l^2 = 2l$$
$$l^2 - 2l = 0$$
$$l(l-2) = 0$$

either $l = 0$ or $l - 2 = 0$ i.e $l = 2$.

Que 10 if a_n is sequence of positive term then

a) $\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} \ge \overline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n}$

b) $\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} \ge \overline{\lim}_{n\to\infty} (a_n)^{1/n}$

c) $\lim_{n\to\infty} (a_n)^{1/n} \leq \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}$

[✓ d)] $\overline{\lim_{n\to\infty}} (a_n)^{1/n} \leq \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

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Q→11 Consider the seqn 4, 0, 4.1, 0, 4.11, 0, 4.111, 0, ---
then

a) Converges to $4\frac{1}{9}$
b) is divergent
[✓ c)] is Not cgt.
d) is Not cgt and has supremum $4\frac{1}{9}$
e) is Unbounded.

Que 12 $\langle a_n \rangle$, $\langle b_n \rangle$, $\langle c_n \rangle$ be seqn of Real Numbers such that $b_n = a_{2n}$, & $c_n = a_{2n+1}$. Then $\langle a_n \rangle$ is cgt.

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a) implies $\langle b_n \rangle$ is cgt but $\langle c_n \rangle$ Need Not be cgt
b) implies $\langle c_n \rangle$ is cgt but $\langle b_n \rangle$ Need Not be cgt
[✓ c)] implies both $\langle b_n \rangle$ and $\langle c_n \rangle$ are cgt.
d) implies both $\langle b_n \rangle$ and $\langle c_n \rangle$ are divergent.

Que 13 $\langle a_n \rangle = \langle \sqrt{n+1} - \sqrt{n} \rangle$ then limit Superior and limit Inferior of an are.

a) $\sqrt{2}+1$ and 0

b) $\frac{1}{\sqrt{2}+1}$ and 0

[✓ c)] both equal to 0

d) both equal to $\frac{1}{\sqrt{2}+1}$

Soln :- $\langle a_n \rangle = \langle \sqrt{n+1}-\sqrt{n} \rangle$

$$a_n = \frac{\sqrt{n+1}-\sqrt{n} \times \sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

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$$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\lim_{n\to\infty} = 0$$

i.e both equal to 0

Que14 Let $A = \{1, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, ---\}$

is Range of seqn $\langle a_n \rangle$. Then set of limit point of A

a) $[0,1]$

[✓ b)] $[0,1] \cap \mathbb{Q}$

c) $[0,1] \cap \frac{\mathbb{R}}{\mathbb{Q}}$

d) Empty set.

Que15 Let $\langle a_n \rangle$ is a Non Negative Real Numbers seqn. Set $\sum_{n=1}^{\infty} a_n$ is cgt. if P is a real Number set

$\sum \frac{\sqrt{a_n}}{n^P}$ diverges then.

a) $P \in (0,1)$, (b) $P \in [0,1]$
c) $P \in [0,1)$, d) $P \in (0,1]$

Que 16 if $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\dots+\frac{1}{(2n)^2}\right]$ then

a is

ans a) 0 c) $1/e$
b) 1 d) e

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Sol^n $a_n = \frac{1}{4n}$

$\lim_{n\to\infty} a_n = 0$.

$\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+ - - + \frac{1}{(2n)^2}\right] = 0$.

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Que 17 $\lim_{n\to\infty}\left[\frac{1-2+3-4+5-6+\dots+(-2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}\right]$ equals

a) ∞ b) $1/2$
c) 0 d) $-1/2$

Q-18 Choose correct statements.

a) A sequence $\langle a_n \rangle \longrightarrow 0$ iff $|\langle a_n \rangle| \rightarrow 0$

b) if sequence $\langle a_n \rangle \longrightarrow 0 \Rightarrow |\langle a_n \rangle| \longrightarrow 0$ but Converse may Not be true.

c) if $\langle b_n \rangle \rightarrow 0$ and $|a_n| \leq |b_n| \ \forall \ n$ then $\langle a_n \rangle$ is convergent but may Not be converges to 0.

d) if $\langle b_n \rangle \longrightarrow 0$ and $|a_n| \leq |b_n| \ \forall \ n$ then $\langle a_n \rangle$ must converges to zero.

Q-19 Let $f : (a,b) \longrightarrow \mathbb{R}$ be a cts function and $\langle a_n \rangle$ be a sequence in (a,b) then

a) if $\langle a_n \rangle$ is cauchy seq^n then $\langle f(a_n) \rangle$ is convergent Sequence.

b) $\langle f(a_n) \rangle$ is bounded sequence.

c) if $\langle a_n \rangle$ is Monotonically $\downarrow$ then $\langle f(a_n) \rangle$ is also Monotonically $\downarrow$.

d) if $\langle a_n \rangle$ is convergent seq^n then $\langle f(a_n) \rangle$ is Bounded sequence.

Q-20 Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log$, then

a) $\langle x_n \rangle$ is increasing and $x_n \geq 0 \ \forall \ n \in \mathbb{N}$

b) $\langle x_n \rangle$ is decreasing and $x_n \leq 0 \ \forall \ n \in \mathbb{N}$

c) $\langle x_n \rangle$ is increasing and $x_n \leq 0 \ \forall \ n \in \mathbb{N}$

d) $\langle x_n \rangle$ is decreasing and $x_n \geq 0 \ \forall \ n \in \mathbb{N}$

Que 21 If $\langle a_n \rangle$ be a seq^n. s.t $\limsup \langle a_n \rangle = a$ and $\lim\inf \langle a_n \rangle = b$ where $a, b \in \mathbb{R}$. Then which of the following statement is / are correct.

1) $\langle a_n \rangle$ oscillate finitely if $a = b$
2) $\langle a_n \rangle$ oscillate Infinitely if $a = b$
3) $\langle a_n \rangle$ is cgt if $a = b$

Code

a) 1 and 2 only
b) 1 and 3 only.
c) 1 only
d) None to these.

Que 22 $\langle a_n \rangle$, $\langle b_n \rangle$ are two sequences, then choose the correct answer.

a) If $\langle a_n + b_n \rangle$ and $\langle a_n b_n \rangle$ both cgt then $\langle a_n \rangle$ and $\langle b_n \rangle$ are cgt.

b) If $\langle a_n - b_n \rangle$ and $\left\langle \frac{a_n}{b_n} \right\rangle$ both cgt then $\langle a_n \rangle$ and $\langle b_n \rangle$ are cgt

c) If $\langle a_n + b_n \rangle$ and $\langle a_n - b_n \rangle$ both cgt then $\langle a_n \rangle$ & $\langle b_n \rangle$ are cgt.

d) None of these.

Que 23 $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$, then.

a) $\langle a_n \rangle$ converges to 0
b) $\langle a_n \rangle$ converges to 1
c) $\langle a_n \rangle$ diverges to ∞
d) None of above.

Que-24 let $\langle x_n \rangle$ & $\langle y_n \rangle$ are cgt and Monotonic seq^n Respectively then $\langle x_n \cdot y_n \rangle$

a) Always cgt
b) cgt if $\langle x_n \rangle$ is Monotonic
c) converges if $\langle x_n \cdot y_n \rangle$ is Monotonic
d) converges if $\langle y_n \rangle$ is bdd.

Que-25 $\lim_{n \to \infty} \left[\frac{(n+1) \cdot (n+2) \cdots (n+n)}{n^n} \right]^{1/n}$ is

a) $\frac{4}{e}$ b) $\frac{3}{e}$

c) $\frac{2}{e}$ d) $\frac{1}{e}$

Que 26 Let $\langle a_n \rangle$, $\langle b_n \rangle$, $\langle c_n \rangle$ be seq^n of Real Numbers s.t $b_n = a_{2n}$, $c_n = a_{2n+1}$ and $\langle a_n \rangle$ is cgt seq^n, then which of the following are Not true?

a) $\langle b_n \rangle$ converges but $\langle c_n \rangle$ Need Not be
b) $\langle c_n \rangle$ converges but $\langle b_n \rangle$ Need Not be.
c) Both $\langle b_n \rangle$ and $\langle c_n \rangle$ converge but $\lim_{n\to\infty} b_n \neq \lim_{n\to\infty} c_n$.
d) Both $\langle b_n \rangle$ and $\langle c_n \rangle$ are divergent.

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Que→27 If terms of this oscillating series are grouped pairwise $3 - \frac{2^2+1}{2} + \frac{2^3+1}{2^2} - \frac{2^4+1}{2^3} + \dots$, then Resulting series.

a) converges
b) Diverges
c) oscillates infinitely
d) oscillates finitely.

Que 28 consider the following statements is / are correct

a) Every convergent seq^n is a cauchy sequence
b) Every cauchy seq^n of Rational numbers is conv.
c) Every cauchy seq^n of Real Number is conv.
d) Every convergent seq^n is Bounded seq^n.

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? Que 29 if $\langle a_n \rangle$ has finite Number of Limit points then.

a) $\langle a_n \rangle$ is bdd

b) $\langle a_n \rangle$ is oscillatory.

c) $\langle a_n \rangle$ is Bounded atleast from one side

d) None of these.

Qus 30 A cauchy sequence in $\mathbb{Q}$ which doesn't have a limit in $\mathbb{Q}$.

a) $\left\{1, 2+\frac{1}{2}, 3+\frac{1}{3}, 4+\frac{1}{4}, ------\right\}$

b) $\left\{1, 1+\frac{1}{1!}, 1+\frac{1}{1!}+\frac{1}{2!}, 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}, ------\right\}$

c) $\left\{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ------\right\}$

d) $\left\{1, -1, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2^2}, \frac{-1}{2^2}, ------\right\}$

2011
I.IT Bombay.

Qus 31 Let $\langle x_n \rangle$ monotonically increasing sequence of Real No. and let $\langle x_n \rangle$ be bdd then

a) $\langle x_n \rangle$ is always convergent

b) $\langle x_n \rangle$ is cgt only if $x_n \geq 0$

c) $\langle x_n \rangle$ is a cauchy seqn but Not cgt.

d) $\langle x_n \rangle$ Need Not ~~to~~ be cauchy seqn.

IIT Bombay.
2009

Qus 32 Let $y_1 = 1$ and $y_{n+1} = \frac{3y_n + 2}{6}$, $n \geq 1$. Then the seq^n $\langle y_n \rangle$.

a) converges to 0
b) Converges to 1/2
c) converges to 2/3
d) doesnot converge.

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$\langle y_n \rangle = \{1, \frac{5}{6}, 2, \frac{4}{3}, 1, \frac{5}{6}, 2, \frac{4}{3}\}$

is finite oscillation.
i.e does Not converge.

~~IIT Bombay~~
2007 Qu 33. Let $\{[a_n, b_n]\}$ be a seq^n of Non-degenerate closed intervals in $\mathbb{R}$ satisfying $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$. let

$K = \bigcap_{n=1}^{\infty} [a_n, b_n]$ Which of the following statements is Not appropriate concerning the set K.

a) K is an open set
b) K is empty set.
c) K is Non-empty set.
d) K is singlton set.

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$[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset \cdots \subset [a_2, b_2] \subset [a_1, b_1]$.

$\bigcap_{n=1}^{\infty} [a_n, b_n] = [a_{n+1}, b_{n+1}]$. i.e Non empty set.

NET

Ques34 Which of the following Statements is true.

a) $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim_{x\to\infty} \frac{\log x}{x} = \infty$

b) $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = \infty$ and $\lim_{x\to\infty} \frac{\log x}{x} = 0$

[✓ c)] $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim_{x\to\infty} \frac{\log x}{x} = 0$

d) $\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ but $\lim_{x\to\infty} \frac{\log x}{x}$ doesn't exist.

Ques35 $\langle a_n \rangle$ be a seq^n which is Not Cauchy, then

a) $\langle x_n \rangle$ is Necessarily unbdd.
b) $\langle x_n \rangle$ may be cgt.
c) $\forall\ \epsilon > 0\ \exists$ subseq^n $\langle x_{n_k} \rangle$ s.t. $|x_{n_k} - x_{n_j}| < \epsilon$ for all k and j sufficiently large; $k \neq j$
d) for some $\epsilon > 0\ \exists$ a subseq^n $\langle x_{n_k} \rangle$ s.t. $|x_{n_k} - x_{n_j}| < \epsilon\ \forall\ k\ \&\ j$ Sufficiently large; $k \neq j$

Q→36 Let $\langle x_n \rangle$ be an unbdd seq^n of Non-zero Real No. Then

a) $\langle x_n \rangle$ must have a cgt subsequence.
b) $\langle x_n \rangle$ can't have a convergent subsequence.
[✓ c)] $\left\langle \frac{1}{x_n} \right\rangle$ must have a convergent subsequence.
d) $\left\langle \frac{1}{x_n} \right\rangle$ can't have cgt subsequence

Ques37 ⟨xn⟩ be a seqⁿ of Real Numbers so that

$\sum_{1}^{\infty} |x_n - x| = c$ with 'c' finite, then

a) ⟨xn⟩ may Not be bdd

b) ⟨xn⟩ must be cg. to x

c) ⟨xn⟩ must converge to x+c

d) ⟨xn⟩ is Bounded but Not Necessarily convergent

Ques38 Assume a>1. Then $\lim_{n\to\infty} n^{-2} e^{(\log(n))^a}$ is

a) 0 ; ∀ a>1

b) 0 ; iff 1<a<2

c) ∞ ; ∀ a>1

d) ∞ ; iff 1<a<2

Ques39 Let ⟨xn⟩ be a sequence of Real Numbers Suppose that for each ε>0, there is a Sub-seqⁿ ⟨x_{n_k}⟩ so that $x_{n_k} < x+\epsilon$ ∀ k≥1

Then we must have

a) lim sup $x_n \le x$ b) lim inf $x_n \le x$

c) lim sup $x_n \ge x$ d) lim inf $x_n \ge x$

Ques40. Let A_n be the seqⁿ of intervals

$A_n = \left(1 + \frac{(-1)^n}{n}, 1 + \frac{2}{n}\right)$ for n≥1 then

$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$ is

[✓ a>] Empty set   b> $\{1\}$
c> $(0,3)$ d> $(0,1)$

Q→41 consider the seq^n $\langle x_n\rangle_{n\geq 1}$ defined as
as $x_1 = 1$

$x_n = \sup\{x \in [0, x_{n-1}] : \sin\left(\frac{1}{x}\right) = 0\}$ $n \geq 2$
then $\limsup_{n\to\infty} x_n$ equals.

a) $-\infty$ b> 0
c) 1 d> ∞.

Ques 42 $\langle x_n\rangle$ defined by $x_n = \frac{[nx]}{n}$ for $x \in \mathbb{R}$ where $[\cdot]$ denotes the integer part, then $\langle x_n\rangle$

a> converges to x
b> converges but Not to x
[✓ c>] does Not converge
d> oscillates.

Ques 43 Let $\lim_{x\to 0} x \cdot \sin\frac{1}{x^2}$ equals.

a> 1   [✓ b>] 0   c> oscillates   d> ∞

Date /68

Ques 44 The seq^n $\sqrt{7}$, $\sqrt{7+\sqrt{7}}$, $\sqrt{7+\sqrt{7+\sqrt{7}}}$, ... converges to

a) $\frac{1+\sqrt{33}}{2}$ b) $\frac{1+\sqrt{32}}{2}$

c) $\frac{1+\sqrt{30}}{2}$ [✓ d) $\frac{1+\sqrt{29}}{2}$]

Ques 45 Define $<x_n>$ as $x_1 = 0.1$, $x_2 = 0.101$, $x_3 = 0.101001$, ... Then $<x_n>$

a) converges to a Rational No.
b) converges to an irrational No.
[✓ c) Does Not converge]
d) Oscillates.

Ques 46 Evaluate $\lim_{n\to\infty} n \cdot \sin(2\pi e n!)$

Ques 47 Evaluate $\lim_{\theta\to\pi/2} (1-5\cot(\theta))^{\tan\theta}$

Ques 48 $<a_n>$, $<b_n>$ are convergent sequences of Real Numbers such that $a_n > 0$, $b_n > 0$ ∀ n

Suppose $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ Let $c_n = \frac{a_n}{b_n}$ Then

a) $<c_n>$ converges if $b > 0$
b) $<c_n>$ cgs only if $a = 0$
[✓ c) $<c_n>$ cgs only if $b > 0$]
d) $\lim_{n\to\infty} \sup c_n = \infty$ if $b = 0$

Ques:49 Let $\langle x_n \rangle$ be a sequence of Non-zero real Numbers. Then

a) if $\langle x_n \rangle \to a$ then $a = \sup \langle x_n \rangle$

b) if $\lim \frac{x_{n+1}}{x_n} < 1 \ \forall \ n$ then $\langle x_n \rangle \to 0$

c) if $\langle x_n \rangle < n \ \forall \ n$ then $\langle x_n \rangle$ diverges

d) if $n \le x_n \ \forall \ n$ then $\langle x_n \rangle$ diverges.

Ques:50 $\langle x_n \rangle$, $\langle y_n \rangle$ be two sequences of Real Numbers s.t $x_n \le y_n \le x_{n+1}$, $n = 1, 2, 3, ----$

a) $\langle y_n \rangle$ is a Bounded sequence.

b) $\langle x_n \rangle$ is ↑ sequence

c) $\langle x_n \rangle$ and $\langle y_n \rangle$ converge together

d) $\langle y_n \rangle$ is ↑ sequence.

Ques:51) $\langle x_n \rangle$, $\langle y_n \rangle$ be two sequence of Real Numbers. Find True and False Statements.

a) if $\langle x_n \cdot y_n \rangle$ converges and if $\langle y_n \rangle$ is convergent then $\langle x_n \rangle$ is convergent

b) $\langle x_n + y_n \rangle \to x + y$ if $\langle x_n \rangle \to x$ & $\langle y_n \rangle \to y$.

c) if $\left\langle \frac{x_n}{y_n} \right\rangle$ converges then both $\langle x_n \rangle$ and $\langle y_n \rangle$ Converges

d) if $\langle x_n \rangle$ converges and $\langle y_n \rangle$ diverges then $\langle x_n \cdot y_n \rangle$ diverges.

Que:-5> For any two sequences $<a_n>$ and $<b_n>$

a) $\limsup (a_n+b_n) = \limsup a_n + \limsup b_n$

b) $\limsup (a_n+b_n) \leq \limsup a_n + \limsup b_n$

c) $\limsup (a_n+b_n) \geq \limsup a_n + \limsup b_n$

d) There is No Relation b/w $\limsup (a_n+b_n)$, $\limsup a_n$ and $\limsup b_n$

Que:53> let $<a_n>$ be a seqn of positive terms.

a) $\lim\limits_{n\to\infty} (a_n)^{1/n} < 1 \Rightarrow \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

b) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \lim\limits_{n\to\infty} (a_n)^{1/n} < 1$

c) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Leftrightarrow \lim\limits_{n\to\infty} (a_n)^{1/n} < 1$

d) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ may exist but $\lim\limits_{n\to\infty} (a_n)^{1/n}$ may Not.

Que:54> if $<a_n>$ seqn of Real Numbers Satisfying

$a_{n+1}^2 = 2a_n - 1$ then

a) $\lim\limits_{n\to\infty} a_n = 2$

b) $\lim\limits_{n\to\infty} a_n = 1$

c) $\lim\limits_{n\to\infty} a_n$ does Not exist

d) Oscillate Bounded.

$l^2 = 2l - 1$

$l^2 - 2l + 1 = 0$

$l = 1$

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Ques 53> Which of the following is correct.

[✓ a>] if $\langle a_n \rangle$ converges, every Subseqn of $\langle a_n \rangle$ converg

b> if $\langle a_n \rangle$ doesn't converge, No subsequence of it converges.

c> if a Subseqn of $\langle a_n \rangle$ converges then $\langle a_n \rangle$ also cg

d> if Some Subseqn of $\langle a_n \rangle$ does Not cgs, then No subseqn of $\langle a_n \rangle$ converges.

Q→56> Consider the seqn $a_n = \sin(n)$

a> $\langle a_n \rangle$ is convergent   b> $\liminf_{n\to\infty} \sin(n) = 0$

c> $\langle a_n \rangle$ is divergent to $+\infty$ [✓ d>] $\liminf_{n\to\infty} \sin(n) = -1$

Ques 57> if $\langle a_n \rangle$ be a seqn of Real Number with $\limsup_{n\to\infty} a_n = +\infty$ then

[✓ a>] Set of values taken by $\langle a_n \rangle$ is Not a finite set.

b> $\langle a_n \rangle$ is bdd seq set

c> $\langle a_n \rangle$ can't diverge to ∞.

d> $\langle a_n \rangle$ is convergent

Ques 58> Which of the following is true.

a) a_{2n} cgs then a_n is convergent

[✓ b>] a_{2n} cgs then a_{4n} is convergent

c> a_{2n} is convergent then a_n is Monotonic

d> a_n is Monotonic then a_n is cgt.

Ques 59) $\varlimsup \left\{ \frac{(-1)^n}{2^n} ; n = 1, 2, \ldots \right\}$ is

[✓ a> 0] c> 1

b> 1/4 d> −1

Ques 60) Let $a_n = n^2 + \frac{(-1)^n}{n}$; $n = 1, 2, \ldots$ then

a> Convergent but Not cauchy

b> is cauchy but Not Convergent

c> is Not cauchy But Bounded

[✓ d> does Not have a cgt subseq^n.]

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Ques 61) Let $a_n = (-1)^n \cdot \sin\left(\frac{n\pi}{2}\right)$; $n = 1, 2, \ldots$ then

[✓ a> $\varliminf a_n = -1$ & $\varlimsup a_n = 1$]

b> $\varliminf a_n = 0$ & $\varlimsup a_n = 1$

c> $\varliminf a_n = -1$ & $\varlimsup a_n = 0$

d> $\varliminf a_n = 1$ & $\varlimsup a_n = -1$

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Ques 62) Let $a_n = \frac{4^{3n}}{3^{4n}}$, $n = 1, 2, \ldots$ then $\langle a_n \rangle$ is

a> unbdd b> Bounded But Not convergent

[✓ c> cgt to 0] d> cgt to 1

Ques 63) $\langle a_n \rangle = \cos^2\left(\frac{n\pi}{7}\right) + \sin^3\left(\frac{n\pi}{7}\right) + 1$ then

a) It has cgt subseq^n.
b) It has no cgt subseq^n
c) It has a limit 1
d) It converges to 0

Qu. 64) $\langle a_n \rangle = \frac{3^n}{n^3} - \frac{2^n}{n!}$ then

[✓ a)] $\lim a_n = 0$ b) $\limsup a_n \neq \liminf a_n$
b) $\limsup a_n = +\infty$ d) $\liminf a_n = -\infty$

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Qu. 65) ~~Let $\{x_n\}$~~
$\lim \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{2n}+\sqrt{2n+2}}\right)$ is

a) $\sqrt{2}$ [✓ 2)] $\frac{1}{\sqrt{2}}$

3) $\sqrt{2}+1$ 4) $\frac{1}{\sqrt{2}+1}$

$$\frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{2}+\sqrt{4}} \times \frac{\sqrt{2}-\sqrt{4}}{\sqrt{2}-\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{6}} \times \frac{\sqrt{4}-\sqrt{6}}{\sqrt{4}-\sqrt{6}} + \cdots + \frac{1}{\sqrt{2n}+\sqrt{2n+2}} \times \frac{\sqrt{2n}-\sqrt{2n+2}}{\sqrt{2n}-\sqrt{2n+2}}\right.$$

$$\frac{1}{\sqrt{n}}\left[\frac{\sqrt{2}-\sqrt{4}}{-2} + \frac{\sqrt{4}-\sqrt{5}}{-2} + \cdots + \frac{\sqrt{2n}-\sqrt{2n+2}}{-2}\right]$$

$$\frac{1}{-2\sqrt{n}}\left[\sqrt{2}-\sqrt{2n+2}\right]$$

$$\frac{1}{2\sqrt{n}}\left[\sqrt{2n+2}-\sqrt{2}\right] = \lim \frac{\sqrt{2n}}{2\sqrt{n}}\left[\sqrt{2+\frac{2}{n}} - \frac{\sqrt{2}}{\sqrt{n}}\right] = \frac{1}{\sqrt{2}} \quad \underline{\text{Ans}}$$

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SERIES OF REAL NUMBERS

Series :- Sum of the terms of seq^n is called series

Let $\langle u_n \rangle$ be seq^n of Real Number

i'e $\{u_1, u_2, \ldots, u_n, \ldots\}$

We construct a New seq^n S_n

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i.e $S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

$\vdots$

$S_n = u_1 + u_2 + \cdots + u_n$

$\vdots$

$\langle S_n \rangle$ be Sequence of partial sum i'e S.O.P.S

1) if seq^n $\langle S_n \rangle$ is convergent then series is convergent
2) if seq^n $\langle S_n \rangle$ is divergent then Series is divergent.
3) if seq^n $\langle S_n \rangle$ div. to $+\infty$ $(-\infty)$ then series is div. to $+\infty$ $(-\infty)$

Que. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$S_1 = \frac{1}{1\cdot 2}$

$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$

|
|

$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

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$S_n = \left\{1 - \frac{1}{n+1}\right\}$

S_n is cgt.

$\Rightarrow$ Series is convergent.

Series of Positive terms :- if all the terms of the series $\sum a_n = a_1 + a_2 + \cdots + a_n + \cdots$ are positive i.e $a_n > 0 \ \forall\, n$, Then Series $\sum a_n$ is called positive terms series.

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Alternating Series :- A series in which terms are alternately positive and negative is called an alternating series.

Eg:- $\sum (-1)^{n-1} a_n = a_1 - a_2 + a_3 + \cdots + (-1)^{n-1} a_n + \cdots$

Examples

1) $\langle a_n \rangle = \frac{1}{n}$

$\sum a_n = 1 + \frac{1}{2} + \frac{1}{3} + --- + \frac{1}{n} + ---$

So sum of n terms $= 1 + \frac{1}{2} + \quad + \frac{1}{n}$ which is Not cauchy so $\langle S_n \rangle$ is Not cgt. Hence $\sum a_n$ is divergent.

2) $\langle a_n \rangle = \frac{1}{2^n}$

then $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \quad + \frac{1}{2^n} + ---$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{2^2}$

$S_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$

$\vdots$

$S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + --- + \frac{1}{2^n}$

$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + --- + \frac{1}{2^{n-1}}\right)$

$S_n = \frac{1}{2} \cdot \left[\frac{1 - 1/2^n}{1 - 1/2}\right]$ $\qquad \left\{ \because S_n = \frac{a(1-r^n)}{1-r} \right.$

$= 1 - \frac{1}{2^n}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{2^n}\right) = 1$

$\Rightarrow \langle S_n \rangle \to 1$ as $n \to \infty$.

Hence S.O.P.S is cgt & converges to 1

$\Rightarrow \sum \frac{1}{2^n}$ is cgt and converges to 1.

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4> Series $1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots$ to ∞. Then $\sum a_n$ is divergent.

Soln The nth term of its S.O.P.S is

$$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$= \frac{n}{6}(n+1)(2n+1)$$

So $S_n \to \infty$ as $n \to \infty$

Hence $\langle S_n \rangle$ is divergent

$\Rightarrow \sum a_n$ is divergent to $+\infty$.

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5> $\sum a_n = -1-2-3-\cdots-n-\cdots$ to ∞ then $\sum a_n$ is Divergent and diverges to $-\infty$.

Soln:- $S_n = -1-2-3-\cdots-n$

$$= \frac{-(n)(n+1)}{2}$$

So $S_n \to \infty$ as $n \to \infty$. $\Rightarrow \sum a_n$ is divergent and diverges to $-\infty$.

6) $\sum_{n=0}^{\infty} (-1)^{n+1}$, test for convergence or divergence or oscillates.

Soln - nth term of seqn $\langle S_n \rangle$ will be

$S_n = 1-1+1-1+1-1+ - - -$ to n times

$$S_n \begin{cases} 1 ; & n \text{ is odd} \\ 0 ; & n \text{ is even} \end{cases}$$

The subseq n $\langle S_{2n-1} \rangle$ & $\langle S_{2n} \rangle$ of $\langle S_n \rangle$ converges to 1 and 0 Respectively

$\Rightarrow \langle S_n \rangle$ is Not cgt.

$\langle S_n \rangle$ is bdd & Not cgt $\Rightarrow \langle S_n \rangle$ oscillates finitely.

$\Rightarrow \sum (-1)^{n+1}$ oscillates finitely.

Article The geometric series $1+x+x^2+ - - - -$ to ∞

i) converges if $|x| < 1$
ii) Diverges if $x \geq 1$
iii) oscillates finitely if $x = -1$
iv) oscillates infinitely if $x < -1$

Note:- A geometric series is cgt only it's common Ration r is Numerically less than 1 i.e $|r| < 1$

Note : $\boxed{\lim_{n \to \infty} r^n = 0 \quad \text{if } |r| < 1}$

Note:- Series of positive Number either converge or diverge Never oscillate.

Note:- $\sum a_n$ is convergent iff $<S_n>$ is Bounded above, Here $a_n \geq 0$ $\forall$ $n \in \mathbb{N}$ i.e series of +ve Real Numbers.

i.e Necessary and sufficient condition for a cgt of +ve term series $\sum a_n$ is that its S.O.P.S is bdd above.

Thm:- if the series $\sum a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Note:- Converse of the above theorem is Not true.

Eg:- The series $1+\frac{1}{2}+\frac{1}{3}+ \text{———} +\frac{1}{n}+ \text{———}$ diverges

but $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

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Note:- if $\lim_{n\to\infty} a_n = 0 \Rightarrow \sum a_n$ may or may Not cgt.

Note:- if $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ is Not cgt.

Eg:- $\sum \cos\frac{1}{n}$ so $\lim_{n\to\infty} \cos\frac{1}{n} \neq 0$ so

$\sum \cos 1/n$ not cgt.

Thm:- Let $<u_n>$ be series of +ve Real No. and sequence Monotonic decreasing and converge then

$\lim_{n\to\infty} n u_n = 0$

Note:- But converse of this th^m is Not true.

Ex: The series $\sum \frac{1}{n^2}$

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Note:- if $\lim_{n\to\infty} n u_n = 0 \Rightarrow \sum u_n$ may or may Not converge

Note:- if $\lim_{n\to\infty} n u_n \neq 0 \Rightarrow \sum u_n$ is Not cgt.

Ex:- $\sum \frac{1}{n}$

$\lim_{n\to\infty} n\cdot\frac{1}{n} = 1 \neq 0 \Rightarrow$ Series is Not cgt.

Ex:- $\sum \cos\frac{1}{n} \Rightarrow$ $\lim_{n\to\infty} \cancel{n}\cdot\cos\frac{1}{n} = 1 \neq 0 \Rightarrow$ Series is Not cgt.

Cauchy's General Principle of convergence for Series.

Th^m:- A Series $\sum a_n$ converges iff for each $\epsilon > 0$ there exist a +ve integer m s.t.

$|a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \epsilon \quad \forall\ n \geq m$ and $p \geq 1$

Th^m:- if $\sum a_n = a$ then $\sum c\cdot a_n = c\cdot a$

Th^m:- if $\sum a_n = a$ then $\sum_{n=0}^{\infty} a_n = a + a_0$ and

$\sum_{n=2}^{\infty} a_n = a - a_1$

Thm:- If $\sum a_n = a$ and $\sum b_n = b$ then

1> $\sum (a_n+b_n) = a+b$

2> $\sum (a_n-b_n) = a-b$.

Comparision Tests for Positive term series:-

1> If $\sum a_n$ and $\sum b_n$ are two positive term series, and $k \neq 0$, a fixed positive Real Number (independent of n) and there exist a positive integer m s.t $a_n \leq k b_n$ ∀ $n \geq m$ then

1> If $\sum b_n$ is convergent then $\sum a_n$ is convergent.
2> If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

Otherwise we can write above thm as -
Let $\sum a_n$ and $\sum b_n$ are two positive term series and $a_n \leq b_n$ ∀ $n \in \mathbb{N}$ then

1> $\sum b_n$ converges $\Rightarrow \sum a_n$ converges
2> $\sum a_n$ Diverges $\Rightarrow \sum b_n$ diverges.

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Ex:- $\sum a_n = \sum \sin\frac{1}{n^2}$ is cgt.

Soln: As I know $|\sin x| \leq x$ ∀ $x \in \mathbb{R}$

So $\sum \sin\frac{1}{n^2} \leq \sum \frac{1}{n^2}$

Here $\sum 1/n^2$ is cgt (by P-Series test)

$\Rightarrow \sum \sin 1/n^2$ is cgt.

Ex - $\sum a_n = \sum \left[(1+n^4)^{1/2} - n^2 \right]$

Soln - $(1+n^4)^{1/2} - n^2 = \dfrac{\left[(1+n^4)^{1/2} - n^2\right] \times \left[(1+n^4)^{1/2} + n^2\right]}{\left[(1+n^4)^{1/2} + n^2\right]}$

$$= \frac{1}{(1+n^4)^{1/2} + n^2} < \frac{1}{n^2}$$

i.e $(1+n^4)^{1/2} - n^2 < \dfrac{1}{n^2}$

So $\sum \left[(1+n^4)^{1/2} - n^2\right] < \sum \dfrac{1}{n^2}$

Here $\sum \frac{1}{n^2}$ is cgt so $\sum \left[(1+n^4)^{1/2} - n^2\right]$ is cgt

Ex:- $\sum a_n = \sum e^{-n^2}$ is cgt.

Soln:- As we know $e^{n^2} > n^2$

$$\Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2}$$

$$\Rightarrow e^{-n^2} < \frac{1}{n^2}$$

$$\Rightarrow \sum e^{-n^2} < \sum \frac{1}{n^2}$$

Here $\sum e^{-n^2}$ is cgt bcoz $\sum \frac{1}{n^2}$ is cgt.

2) Limit form Test :- if $\sum a_n$ and $\sum b_n$ are two Positive term series such that $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = l$ [finite and Non-zero], then both behaves alike i.e both converge or diverge together. (depend of behaviour of v_n).

Ex:- $\sum a_n = \sum \sin 1/n$ is divergent.

Soln:- Let $\sum b_n = \sum 1/n$

Now $a_n = \sin\frac{1}{n}$ and $b_n = \frac{1}{n}$

Apply limit form test

$$\lim_{n\to\infty}\left(\frac{\sin 1/n}{1/n}\right) = 1 \quad [\text{finite \& Non-zero}].$$

⇒ both behave alike.

⇒ $\sum b_n = \sum 1/n$ is divergent by p-series Test.

⇒ $\sum a_n$ is divergent i.e $\sum \sin 1/n$ is divergent.

Ex:- $\sum \frac{1}{n^{1+1/n}} = \sum a_n$.

let $\sum b_n = \frac{1}{n}$ then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^{1+1/n}}}{1/n}$

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{n^{1+1/n-1}}\right) = \lim_{n\to\infty}\left(\frac{1}{n^{1/n}}\right) = 1 \neq 0$ | finite

then Here $\sum b_n$ is divergent so $\sum a_n$ is divergent.

Ex→ Series $\frac{1\cdot2}{3^2\cdot4^2}+\frac{3\cdot4}{5^2\cdot6^2}+\frac{5\cdot6}{7^2\cdot8^2}+ - - -$

Sol^n:- Given series can be written as by finding n^{th} term

$$\sum a_n = \sum \frac{(2n-1)\cdot(2n)}{(2n+1)^2\,(2n+2)^2}$$

Let $\sum b_n = \sum 1/n^2$ then by limit form test

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\frac{(2n-1)(2n)}{(2n+1)^2(2n+2)^2}}{1/n^2} = \frac{n^2\,(2n-1)(2n)}{(2n+1)^2\,(2n+2)^2}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\frac{n^4\left(2-1/n\right)(2)}{n^4\left(2+1/n\right)^2\left(2+2/n\right)^2} = \frac{2\cdot2}{2^2\cdot2^2}$$

$\rightarrow \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{1}{4} \neq 0$ and finite.

Thus, two series converge or diverge together.
Since $\sum b_n$ is cgt. $\rightarrow$ $\sum a_n$ is cgt.

3> if $\sum a_n$ and $\sum b_n$ are two positive term series and $\exists$ a positive integer m s.t.

$\frac{a_n}{a_{n+1}} \geq \frac{b_n}{b_{n+1}}$ $\forall$ $n \geq m$ then

1) If $\sum b_n$ is cgt then $\sum a_n$ is cgt

2) If $\sum a_n$ is divergent then $\sum b_n$ is divergent

4) $\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+ - - - -$

from given series we can find a_n

$$a_n = \frac{n\text{th term of } 1,3,5, - - -}{n\text{th term of denominator}} = \frac{2n-1}{n\cdot(n+1)(n+2)}$$

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$$a_n = \frac{2-1/n}{n^2(1+1/n)(1+2/n)}$$

Let $b_n = 1/n^2$ then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{2-1/n}{(1+1/n)(1+2/n)}\right) = 2 \neq 0 \text{ \& finite}$$

by limit form test, both series $\sum a_n$ & $\sum b_n$ behave alike.

So $\sum b_n = \sum 1/n^2$ is cgt so $\sum a_n$ is cgt.

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5) $\frac{1}{3^2}\cdot\frac{2}{4^2}+\frac{3}{5^2}\cdot\frac{4}{6^2}+\frac{5}{7^2}\cdot\frac{6}{8^2}+$

Here $a_n = \frac{(2n-1)\cdot(2n)}{(2n+1)^2(2n+2)^2}$

$$= \frac{4n^2(1-1/n)\cdot(1)}{16n^4(1-1/n)^2(1+2/n)^2}$$

$$a_n = \frac{(1-1/n)}{4n^2\,(1+1/n)^2\cdot(1+2/n)^2}$$

Let $\sum b_n = \sum 1/n^2$ then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\frac{(1-1/n)}{4(1+\frac{1}{n})^2(1+\frac{2}{n}}$

$= \frac{1}{4} \neq 0$ & finite.

then by limit form test $\sum a_n$ is cgt.

Theorem:- or P-Series Test :-

A positive term series $\sum 1/n^p$ is cgt iff $p>1$. [Here $\sum 1/n^p$ is called Hyper Harmonic series or p-series]

i.e the series $\sum 1/n^p = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$ to ∞ cgs if $p>1$ and diverges if $p\le 1$.

Some questions Related to above test.

Examine the convergence or divergence of positive term series.

1) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\cdots$ to ∞

Sol:- ~~Let~~ $\sum a_n = \frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+$

$= \left(\frac{3}{5}+\frac{3}{5^3}+\cdots\infty\right)+\left(\frac{4}{5^2}+\frac{4}{5^4}+\cdots\infty\right)$

Let $\sum a_n = \frac{3}{5}+\frac{3}{5^3}+\cdots$ is G.P series with common Ration $\frac{1}{5^2}$ which less than 1.

So $\sum a_n$ is cgt.

similarly $\sum b_n = \frac{4}{5} + \frac{4}{5^4} + \cdots$ is G.P with

$r = \frac{1}{5^2} < 1$ Hence $\sum b_n$ is also cgt.

then $\sum(a_n + b_n) = \frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \cdots \infty$ is also cgt.

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2> $a + b + a^2 + b^2 + a^3 + b^3 + \cdots$ to ∞

$a + b + a^2 + b^2 + a^3 + b^3 + \cdots$

$= (a + a^2 + a^3 + \cdots) + (b + b^2 + b^3 + \cdots)$

$= \sum a_n + \sum b_n$ (say)

Now $\sum a_n$ and $\sum b_n$ are G.P with common Ratio a and b Respectively.

then $\sum a_n$ is cgt if $|a| < 1$ and $\sum b_n$ is cgt if $|b| < 1$

Hence $\sum(a_n + b_n) = a + b + a^2 + b^2 + a^3 + b^3 + \cdots$ is cgt if $|a| < 1$ and $|b| < 1$.

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3> $1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \cdots$ to ∞

$= \sum \frac{1}{n^{4/3}}$ $\{ p = 4/3 > 1 \}$

by P-Series test series is cgt.

4) $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$ to ∞

$$a_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2(1+1/n)}}$$

So $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{1}{2(1+1/n)}} = \frac{1}{\sqrt{2}} \neq 0$

Hence by Necessary condition if $\lim_{n\to\infty} a_n \neq 0 \Rightarrow$

$\sum a_n$ is Not cgt.

Hence given series is Not cgt.

5) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$

$$a_n = \frac{n+1}{n^p} = \frac{n\cdot(1+1/n)}{n^p} = \frac{1+1/n}{n^{p-1}}$$

let $b_n = \frac{1}{n^{p-1}}$ then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

which is Non zero & finite then if $\sum b_n$ is cgt then $\sum a_n$ will be cgt

but $\sum b_n = \sum \frac{1}{n^{p-1}}$ is cgt if $p-1>1 \Rightarrow p>2$ and divergent if $p-1 \leq 1 \Rightarrow p \leq 2$

$\Rightarrow \sum a_n$ is cgt if $p>2$ & divergent if $p \leq 2$.

6) $\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \dots$ to ∞, p, q be a +ve Number

$$a_n = \frac{(n+1)^p}{n^q} = \frac{n^p\left(1+\frac{1}{n}\right)^p}{n^q} = \frac{\left(1+\frac{1}{n}\right)^p}{n^{q-p}}$$

Let $b_n = \frac{1}{n^{q-p}}$ then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 \neq 0$ & finite

$\sum a_n$ is cgt if $q-p>1 \Rightarrow q>1+p$

and $\sum a_n$ is divergent if $q-p\le 1 \Rightarrow q \le 1+p$.

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7) $\sum \frac{1}{\sqrt{n}} \sin\frac{1}{n}$

$$= \sum \frac{1}{\sqrt{n}}\left[\frac{1}{n} - \frac{1}{3!}\cdot\frac{1}{n^3} + \frac{1}{5!}\cdot\frac{1}{n^5} + \dots\right]$$

$$= \sum \frac{1}{n^{3/2}}\left[1 - \frac{1}{3!}\cdot\frac{1}{n^2} + \frac{1}{5!}\cdot\frac{1}{n^4} + \dots\right] \text{ (n common)}$$

Let $b_n = \frac{1}{n^{3/2}}$ then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!}\cdot\frac{1}{n^4} \dots\right] = 1$$

Which is Non zero and finite then by limit form test $\sum b_n$ is convergent becoz $p = \frac{3}{2} > 1$

So $\sum a_n = \sum \frac{1}{\sqrt{n}} \sin\frac{1}{n}$ is convergent

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8) $\sum \tan^{-1}\frac{1}{n} = \sum a_n$

$\sum\left[\frac{1}{n}-\frac{1}{3n^3}+\frac{1}{5n^5}-\cdots\right]$; $\left[\because \tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}\right.$

$$\sum\frac{1}{n}\left[1-\frac{1}{3n^2}+\frac{1}{5n^4}-\cdots\right]=\sum a_n$$

Let $b_n=\frac{1}{n}$ then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=1\neq 0$ & finite

Here $\sum b_n$ is divergent so $\sum a_n$ is divergent.

9) $\frac{1}{\log 2}+\frac{1}{\log 3}+\frac{1}{\log 4}+\cdots$

The Series is $\sum_{n=2}^{\infty}\frac{1}{\log n}$

as know $\log n < n \ \forall\ n\geq 2$

$\frac{1}{\log n} > \frac{1}{n} \ \forall\ n\geq 2$

then by comparision Ist test, if smaller seqn is divergent then larger will be divergent.

Here $\sum\frac{1}{n} < \sum\frac{1}{\log n}\ \forall\ n\geq 2$

and $\sum\frac{1}{n}$ is divergent so $\sum\frac{1}{\log n}$ is divergent.

Notes:- Addition or deletion of a finite Number of terms does Not alter the Nature of the series.

Eg:- $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots$

we are going to find a_n by leaving aside the first term

$$a_n = \frac{n^n}{(n+1)^{n+1}}$$

so after leaving first terms the Nature of the series i'e cgt or divergence will Not be changed.

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Note:- if $\sum a_n^2$ & $\sum b_n^2$ both are convergent then $\sum a_n b_n$ is also convergent.

Note:- if $\sum a_n$ is convergent series of +ve terms then $\sum a_n^2$ is also cgt. converse Not true.

Eg:- $\sum \frac{1}{n^2}$ is cgt where as $\sum 1/n$ is divergent.

Que Test the convergence of series

$$\frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \cdots$$

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Solⁿ:- $a_n = \frac{x^{n-1}}{1+x^n}$

Case I when $0<x<1$

when $0<x<1$ then $x^n \to 0$ as $n \to \infty$

Let $b_n = x^{n-1}$ then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1+x^n}\right) = 1 \neq 0 \text{ \& finite}$$

∴ $\sum a_n$ & $\sum b_n$ behave alike

but $\sum b_n = \sum x^{n-1}$ is G.P with common Ration x Also $0<x<1$ so $\sum b_n$ is cgs ⇒

$\sum a_n$ is convergent.

case-II when $x=1$

$$a_n = \frac{1}{1+1} = \frac{1}{2}$$

then $\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$

⇒ $\sum a_n$ is divergent.

case-III when $x>0$ i.e $0<\frac{1}{x}<1$

when $x>0$ or $0<\frac{1}{x}<1$ then $\frac{1}{x^n} \to 0$ as $n\to\infty$

$$a_n = \frac{x^{n-1}}{1+x^n} = \frac{x^n \cdot x^{-1}}{1+x^n} = \frac{1}{x} \cdot \frac{x^n}{1+x^n} = \frac{1}{x} \cdot \frac{1}{1+1/x^n}$$

Now $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{x} \cdot \frac{1}{1+\frac{1}{x^n}} = \frac{1}{x} \neq 0$

⇒ $\sum a_n$ is divergent.

= In all if $0<x<1$ then $\sum a_n$ is cgt and if $x \geq 1$ then $\sum a_n$ is divergent.

In above Example series is Not Cgt then we are directly saying that Series is divergent because +ve term series either converges or diverge but never oscillates.

1) $\sum \sin\frac{1}{n}$

$u_n = \sin\frac{1}{n} \sim \frac{1}{n} = v_n$

$v_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1$

and $\sum v_n$ is divergent

$\Rightarrow \sum u_n$ is divergent.

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2) $\sum \sin\frac{1}{n^p}$

$u_n = \sin\frac{1}{n^p} \sim \frac{1}{n^p} = v_n$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin 1/n^p}{1/n^p} = 1 \neq 0$ & finite.

$p>1$ $\sum v_n$ is cgt for $p>1$

$\Rightarrow$ $\sum u_n$ is cgt for $p>1$

$\sum v_n$ is divergent for $p\le 1$

$\Rightarrow \sum u_n$ is divergent for $p \le 1$.

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3) $\sum \cos\frac{1}{n^p}$

$a_n = \cos\frac{1}{n^p}$

$\lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$

$\Rightarrow \sum \cos\frac{1}{n^p}$ is divergent if $p>0$

$\Rightarrow \sum \frac{1}{n}\sin\frac{1}{n}$

$u_n = \frac{1}{n}\sin\frac{1}{n} \sim \frac{1}{n^2} = v_n$

$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1/n \,\sin 1/n}{1/n^2} = \lim_{n\to\infty}\frac{\sin 1/n}{1/n} = 1 \neq 0$ & finite

$\Rightarrow \sum v_n$ is cgt

$\Rightarrow \sum u_n$ is cgt.

5) $\sum \frac{1}{n^p}\sin\frac{1}{n}$

$u_n = \frac{1}{n^p}\sin\frac{1}{n} \sim \frac{1}{n^{p+1}} = v_n$

$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1/n^p \,\sin 1/n}{1/n^{p+1}} = \lim_{n\to\infty}\frac{\sin 1/n}{1/n} = 1 \neq 0$ & finite

$\Rightarrow \sum v_n$ is cgt if $p+1>1$ i.e $p>0$

$\Rightarrow \sum u_n$ is cgt if $p>0$

6) $\sum \frac{1}{n}\cos\frac{1}{n}$

$u_n = \frac{1}{n}\cos\frac{1}{n} \sim \frac{1}{n} = v_n$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1/n \cos 1/n}{1/n} = 1 \neq 0$ & finite.

$\sum v_n$ is divergent $\Rightarrow$ $\sum u_n$ is divergent.

7) $\sum \frac{1}{n^p} \cos \frac{1}{n}$

$u_n = \frac{1}{n^p} \cos \frac{1}{n}$ $\frac{1}{n^p} = v_n$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1/n^p \cos 1/n}{1/n^p} = 1 \neq 0$ & finite.

$\sum v_n$ is cgt if $p > 1$

$\Rightarrow \sum u_n$ is cgt if $p > 1$

$\sum v_n$ is divergent if $p \leq 1$

$\Rightarrow \sum u_n$ is divergent if $p \leq 1$, or $0 < p \leq 1$

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Fibonacci Seqn:- Let $<u_n>$ be a seqn of Real Numbers of type $a_{n+2} = a_{n+1} + a_n$ and $a_1 = 1$ and $a_2 = 1$ becomes Fibonacci seqn and diverge to $+\infty$

Contractive seqn:- Let $<x_n>$ be a seqn of Real Numbers is called contractive seqn if $\exists$ a constant c, $0 < c < 1$ s.t

$|x_{n+2} - x_{n+1}| \le c\,|x_{n+1} - x_n|$

where c is called the constant of constractive seq

and contractive seqn is cauchy sequence.

Qus $\sum \frac{1}{n^p}\tan\frac{1}{n}$. Converges if $p>0$, $p>1$

(behaviour as $\sum \frac{1}{n^p}\sin\frac{1}{n}$ in case of limit).

D' Alembert Ratio test :-

if $\sum a_n$ is a positive term series sot

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then the series

i) converges, if $l<1$
ii) Diverges, if $l>1$ and
iii) Test fail, if $l=1$ ie we can't say about the Nature of Series if $l=1$.

Ex- $\sum \frac{1}{n}$ divergent ; $l=1$

$\sum \frac{1}{n^2}$ convergent ; $l=1$.

Cauchy Root test :- if $\sum a_n$ is a positive term series such $\lim_{n\to\infty} (a_n)^{1/n} = l$ then the series

i) converges, if $l<1$
ii) Diverges, if $l>1$
iii) Test fail ; if $l=1$ ie if $l=1$ then series

may be converge or may be diverge,

Eg&:- $\sum \frac{1}{n}$, $\lim_{n\to\infty} (1/n)^{1/n} = 1$, $l=1$

and divergent

$\sum 1/n^2$, $\lim_{n\to\infty} (1/n^2)^{1/n} = 1$, $l=1$

and cgt.

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Most I.M

# ~~Cachy~~ Cauchy Condensation test :- Let $\langle f(n) \rangle$ be seqⁿ of +ve Real Number and seqⁿ is Monotonicaly decreasing then series $\sum f(n)$ and $\sum a^n f(a^n)$ converge or diverge together and $a>1$

(In case of Logarithm Question Apply this test)

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# Raabe's test :-

if $\sum a_n$ is a positive term series such that

$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = l$ then the series

i) Converges ; if $l>1$
ii) Diverges ; if $l<1$ &
iii) Test fails ; if $l=1$

Logarithmic Test :-

if $\sum a_n$ be a positive term Series such tha

$\lim_{n\to\infty}\left(n \log \frac{a_n}{a_{n+1}}\right) = l$, then the series

i) converges ; if $l > 1$

ii) Diverges ; if $l < 1$

Que $\sum a_n = \sum \frac{1}{2^{n-1}+1}$

Here $a_n = \frac{1}{2^{n-1}+1}$, $a_{n+1} = \frac{1}{2^n+1}$

$$\frac{a_n}{a_{n+1}} = \frac{2^n+1}{2^{n-1}+1} = \frac{2^n\left(1+1/2^n\right)}{2^{n-1}\left(1+1/2^{n-1}\right)} = \frac{2\left(1+1/2^n\right)}{\left(1+\frac{1}{2^{n-1}}\right)}$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1/2 < 1$ so $\sum a_n$ is cgt.

Que $\sum a_n = \sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Let $a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{1+\frac{1}{2}}}}$

i.e $a_n = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n \cdot n^{1/2}}}$

Now $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$

Here $l = \frac{1}{e} < 1$ hence $\sum a_n$ is cgt

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Eg:- $\sum a_n = \sum \frac{n^2-1}{n^2+1} \cdot x^n \; ; \; x > 0$

$$a_n = \frac{n^2-1}{n^2+1} \cdot x^n$$

$$a_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1} \cdot x^{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n^2+1)-1\,(n^2+1)}{(n^2+1)+1\,(n^2-1)} \cdot \frac{x^{n+1}}{x^n}$$

$$\lim_{n\to\infty} \frac{(1+1/n^2)-1/n^2}{(1+1/n^2)+1/n^2} \cdot \frac{(1+1/n^2)}{(1-1/n^2)} \cdot x$$

$$= x.$$

Hence by D' Alembert Ratio Test. Series is cgt if $x < 1$ and diverges if $x > 1$

Now for $x = 1$

$a_n = \frac{n^2-1}{n^2+1}$, and $\lim_{n\to\infty} \left(\frac{1-1/n^2}{1+1/n^2} \right) = 1 \neq 0$

at $x = 1$ Series is divergent.

Hence Series is cgt if $x < 1$ and diverges if $x \geq 1$.

Eg:- $\sum a_n = \sum \frac{3 \cdot 6 \cdot 9 \cdots 3n}{7 \cdot 10 \cdot 13 \cdots (3n+4)} \cdot x^n \; ; \; x > 0$

Note :- When Ratio test fail then we should apply Rabbe's test.

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So. $a_n = \dfrac{3\cdot6\cdot9\cdots3n}{7\cdot10\cdot13\cdots(3n+4)}$, $a_{n+1} = \dfrac{3\cdot6\cdot9\cdots3n\cdot(3n+}{7\cdot10\cdot13\cdots(3n+4)(3n}$

$$\frac{a_n}{a_{n+1}} = \frac{3\cdot6\cdot9\cdots(3n+4)\cdot(3n+7)}{7\cdot10\cdot13\cdots(3n+4)}\cdot\frac{3\cdot6\cdot9\cdots3n}{3\cdot6\cdot9\cdots3n\,(3n+3)\,x}$$

$$\frac{a_n}{a_{n+1}} = \frac{3n+7}{(3n+3)}\cdot\frac{1}{x}$$

So $\dfrac{a_{n+1}}{a_n} = \left(\dfrac{3n+3}{3n+7}\right)\cdot x = \left(\dfrac{1+\frac{3}{3n}}{1+\frac{7}{3n}}\right)\cdot x$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = x$$

By Ratio test Σa_n is cgt if $x<1$ & diverges if $x>1$ and Test fail at $x=1$.

Now By Raabe's Test

$$\lim_{n\to\infty} n\cdot\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty} n\cdot\left(\frac{3n+7}{3n+3}\cdot1-1\right)$$

$$= \lim_{n\to\infty} n\cdot\left(\frac{3n+7-3n-3}{3n+3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{4n}{3n+3}\right) = \frac{4}{3} > 1$$

Hence By Raabe's test Σa_n is cgt at $x=1$

In all Σa_n is cgt if $x\le1$ and divergent if $x>1$

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Eg: $\sum a_n = 1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 \cdot x^3}{3!} + \cdots \; ; \; x > 0$

leaving first term

$$a_n = \frac{n^n \cdot x^n}{n!} , \quad a_{n+1} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n \cdot x^n} = \frac{n^{n+1}\left(1 + 1/n\right)^{n+1} \cdot x}{(n+1) \; n^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1} \cdot x}{n^{n+1}\left(1+\frac{1}{n}\right)} = \left(1+\frac{1}{n}\right)^n \cdot x$$

So $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} e^{1/n \times n} \cdot x = \lim\limits_{n\to\infty} e \cdot x$

Hence $\sum a_n$ is cgs if $ex < 1 \Rightarrow x < \frac{1}{e}$ and diverges if $ex > 1 \Rightarrow x > \frac{1}{e}$ & Test fail at $x = \frac{1}{e}$

$$\frac{a_n}{a_{n+1}} = \left(\frac{n}{n+1}\right)^n \cdot e$$

$$\lim_{n\to\infty}\left(n \log \frac{a_n}{a_{n+1}}\right) = \lim_{n\to\infty} n \left(\log\left(\frac{n}{n+1}\right)^n \cdot e\right)$$

$$= \lim_{n\to\infty} n \left[\log e - n \log\left(1 + \frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} n \left[1 - n\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \cdots \right\}\right]$$

$$= \frac{1}{2} < 1$$

by logarithmic test $\sum a_n$ is divergent at $x = \frac{1}{e}$

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52 Cauchy's Integral Test :-

If for $x \geq 1$, $f(x)$ is a Non Negative monoton cally decreasing integrable function of x such th $f(n) = u_n$ for all positive integral value of n, the the series $\sum_{n=1}^{\infty} u_n$ and the improper integral $\int_1^{\infty} f(x)dx$ converges or diverges together.

Convergence of ~~Imop~~ Improper Integral :-

An integral $\int_a^{\infty} f(x)dx$ is said to be improper integral. Then any infinite or improper integral $\int_a^{\infty} u(x)dx$ is said to be convergent if $t(x) = \int_a^{x} u(x)d$ tends to a finite limit as $x \to \infty$, otherwise, integral said to diverge.

Eg:- $\sum \frac{1}{n(n+1)}$.

[we can apply direct Rabbe's Test].

$u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Put $f(x) = \frac{1}{x} - \frac{1}{x+1}$

For $\forall$ $x \geq 1$ $f(x) \geq 0$ and Monotonically decreasing fun. of x. Cauchy integral test applicable.

Let $I_n = \int_1^{n} f(x)dx = \int_1^{n} \left[\frac{1}{x} - \frac{1}{x+1}\right]dx$

$$= \left[\log x - \log(x+1)\right]_1^n = \left[\log \frac{x}{x+1}\right]_1^n$$

$$I_n = \log\frac{n}{n+1} - \log\frac{1}{2}$$

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\left[\log\frac{n}{n+1} - \log\frac{1}{2}\right]$$

$$= \lim_{n\to\infty}\left[\log\frac{1}{1+\frac{1}{n}} - \log\frac{1}{2}\right]$$

$$= \log 1 - \log\frac{1}{2} = \log\left(\frac{1}{\frac{1}{2}}\right)$$

$$\int_1^\infty f(x)\,dx = \log 2 \quad [\text{finite quantety}]$$

$$\Rightarrow \int_1^\infty F(x)\,dx \text{ is cgt} \Rightarrow \sum_{n=1}^{n} u_n \text{ is cgt.}$$

Some Exambles of Series with their Nature.

1) $\sum \frac{1}{(\log n)^{\log n}}$, $\sum \frac{1}{(\log\log n)^{\log n}}$, both cgt.

2) $\sum \frac{1}{n\log n}$, $\sum \frac{1}{n(\log n)(\log\log n)}$, $\sum \frac{1}{n(\log n)(\log\log n)(\log\log\log n)}$.

all three series are divergent.

3) $\sum \frac{1}{n(\log n)^p}$, $\sum \frac{1}{n(\log n)(\log\log n)^p}$,

both cgs if $p>1$ and diverges if $p\le 1$.

Telescopic Series :- Let Σa_n be a series of Real Numbers define b_n and c_n as

$b_n = a_{n+1} - a_n$ and $c_n = a_n - a_{n+1}$

Then Σb_n & Σc_n are telesopic series.

Now partial sum of Σb_n.

$S_n = b_1 + b_2 + - - - + b_n$

$a_2 - a_1 + a_3 - a_2 + a_4 - a_3 + - - + a_{n+1} - a_n$

$\boxed{S_n = a_{n+1} - a_1}$

Paritial Sum of Σc_n

$S_n' = c_1 + c_2 + - - - + c_n$

$= a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + - - - + a_n - a_{n+1}$

$\boxed{S_n' = a_1 - a_{n+1}}$

then $\lim_{n\to\infty} S_n'$ or $\lim_{n\to\infty} S_n$ become finite if $\langle a_n \rangle$ is convergent.

i.e. $\boxed{\Sigma b_n \text{ and } \Sigma c_n \text{ are cgt iff } \langle a_n \rangle \text{ is cgt.}}$

Leibnitz Test for alternating Series.

If the alternating series $a_1 - a_2 + a_3 - a_4 + - - - -$ $(a_n > 0\ \forall n)$ is such that

i) $a_{n+1} < a_n\ \forall n$, and

ii) $\lim_{n\to\infty} a_n = 0$

then the Series converges.

Notes - for Positive Real Number series, we have discussed earlier many tests. But for alternating series we will use only Leibnitz test:

Eg:- $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} + \dots \quad p>0$

here $a_n = \frac{1}{n^p}$ and

i) $a_{n+1} \le a_n \ \forall n$

i.e $\frac{1}{1^p} \ge \frac{1}{2^p} \ge \frac{1}{3^p} \ge \dots$ so on

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$

Hence by Leibnitz test, the given series is converges. i.e

$$\sum \frac{(-1)^{n-1}}{n^p} \text{ is cgt.}$$

Result:- $\sum a_n$ is convergent and $\sum b_n$ is divergent then $\sum(a_n \pm b_n)$ is divergent.

Result:- $\sum a_n$ is divergent and $\sum b_n$ is also divergent then $\sum(a_n \pm b_n)$ have no result i.e it may converge or diverge.

Qu - If $\sum a_n$ is convergent then show that

$\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \ge 0$ is cgt.

Sol^n :- $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$

$a_n + \frac{1}{n^2} - \frac{2}{n}\sqrt{a_n} \geq 0$

$\Rightarrow a_n + \frac{1}{n^2} \geq \frac{2}{n}\sqrt{a_n}$

Here $\sum a_n$ is cgt and $\sum \frac{1}{n^2}$ is cgt

$\Rightarrow \sum a_n + \frac{1}{n^2}$ is cgt, by comparision test $\frac{2}{n}\sqrt{a_n}$ is cgt. In all $a_n + \frac{1}{n^2} - \frac{2}{n}\sqrt{a_n}$ is cgt

i.e $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$ is convergent.

Que $\sum a_n$ is convergent then $\sum \frac{a_n}{a_{n+1}}$ is cgt.

Sol^n :- $\frac{a_n}{a_{n+1}} < a_{n+1}$ by comparision test $\sum \frac{a_n}{a_{n+1}}$ is cgt

Result :- Let seq^n $\langle a_n \rangle$ is a Monotonically decrease and $a_n \geq 0 \ \forall n$ if $\lim_{n\to\infty} a_n = 0$ then the series $\sum (-1)^{n+1} a_n$ is cgt [Leibnitz Rule].

Eg :- $\sum (-1)^n \sin\frac{1}{n}$

Here $a_n = \sin\frac{1}{n}$ which is ↓ and $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \sum (-1)^{n+1} \sin\frac{1}{n}$ is cgt.

Conditional convergence :- Let $\sum a_n$ is a convergent series but $\sum |a_n|$ is divergent, then $\sum a_n$ is said to be conditional cgt series.

|or|

A series which is cgt but is not absolutely cgt is called conditional cgt series.

E.g.:- $\sum (-1)^n \sin\frac{1}{n}$ is conditional convergent

Here $\sum (-1)^n \sin\frac{1}{n}$ is cgt but $\sum \left|(-1)^n \sin\frac{1}{n}\right|$

$= \sum \sin\frac{1}{n}$ is Not cgt.

Absolute convergence :-

Let $\sum a_n$ be a series of mixed and unrestricted sign term s.t $\sum |a_n|$ is convergent then $\sum a_n$ is said to be absolute convergent series.

E.g.:- $\sum (-1)^n \frac{1}{n^2}$ is Absolute convergent series.

$\sum \frac{1}{n^2}$ is cgt.

Note: If series is absolutely convergent then series is cgt Not conversely.

Eg:- $\sum \frac{(-1)^{n-1}}{n}$ Here $\sum \left|\frac{(-1)^{n-1}}{n}\right|$ = divergent But

$\sum \frac{(-1)^{n-1}}{n}$ is convergent

i.e convergent $\not\Rightarrow$ Absolute convergent.

Result from the defination of conditional convergence

Let $\sum a_n$ be a series of mixed term let us defined

$p_n = Max\{a_n, 0\}$ and $q_n = min\{a_n, 0\}$

$p_n = \frac{1}{2}[a_n + |a_n|]$ & $q_n = \frac{1}{2}[a_n - |a_n|]$

a, b two Real Numbers then we have a Result

$Max(a, b) = \frac{1}{2}[a + b + |a-b|]$

$Min(a, b) = \frac{1}{2}[a + b - |a-b|]$

(i) if $\sum a_n$ is absolutely convergent then $\sum p_n$ and $\sum q_n$ is convergent.

(ii) if $\sum a_n$ is conditionally convergent then $\sum p_n$ & $\sum q_n$ is divergent.

Result:- If $\sum a_n$ is an absolutely cgt series and $\langle b_n \rangle$ is bdd seqⁿ then $\sum a_n \cdot b_n$ is Absolutely convergent.

Note:- when series is absolutely convergent then any of Rearrangement of terms doesn't alter the Nature or sum.

Note:- If Series is condinally convergent then any kind of rearrangement of terms may alter the sum, even It can be made divergent.

Note:- In fact if series is conditionally convergent for any real Number α there exist a Rearrangement sot the rearranged series converges to α.

Note:- The sum of Series is the limit of SOPS of corresponding Series. So to find the sum of Series its only SOPS obtain after the Rearrange of series, can give the sum of Series and claim the convergence of Series.

Rearrangement of terms:-

We are going to show how the sum change when we change the Rearrangement of terms of conditional convergent series.

Ex:- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ is conditional convergent Series.

define $\gamma_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n$ [Note].

and $\lim_{n\to\infty}\gamma_n = \gamma$ [Euler's constant].

Now Sum of $2n$ terms of Series.

$$\sum_{n=1}^{2n} a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} - \cdots - \frac{1}{2n}$$

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$$\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$$

$$=\gamma_{2n}+\log 2n-(\log n+\gamma_n)$$

$$=\gamma_{2n}+\log 2+\log n-\gamma_n-\log n$$

$$=\gamma+\log 2-\gamma \quad [\,n\to\infty \ \ \gamma_{2n}\ \&\ \gamma_n\to\gamma\,]$$

$$\sum_{n=1}^{2n} a_n=\log 2$$ (ie sum of 2n terms is log2)

Now we are taking other Rearrangement of n terms.

3. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots(-1)^{n+1}\cdot\frac{1}{n}$

$$=\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$$

Leaving aside first term the last term will be $\frac{1}{4n-1}$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{4n-1}+\frac{1}{4n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$$
$$-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{4n}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{4n-1}+\frac{1}{4n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$
$$-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)$$

$$=\gamma_{4n}+\log 4n)-\frac{1}{2}(\gamma_{2n}+\log 2n)-\frac{1}{2}(\gamma_n+\log n)$$

$$\gamma_{4n}+\log 4+\log n-\frac{1}{2}\gamma_{2n}-\frac{1}{2}\log 2-\frac{1}{2}\log n-\frac{1}{2}\gamma_n-\frac{1}{2}\log n$$

$= \gamma + \log 4 - \frac{1}{2}\gamma - \frac{1}{2}\log 2 - \frac{1}{2}$

$= 2\log 2 - \frac{1}{2}\log 2 = \frac{3}{2}\log 2.$

In general.

$= \gamma_{2pn} + \log 2pn - \frac{1}{2}(\gamma_{pn} + \log pn) - \frac{1}{2}(\gamma_{qn} + \log qn)$

$= \gamma_{2pn} + \log 2 + \log p + \log n - \frac{1}{2}\gamma_{pn} - \frac{1}{2}\log p - \frac{1}{2}\log n$

$- \frac{1}{2}\gamma_{qn} - \frac{1}{2}\log q - \frac{1}{2}\log n$

when $n \to \infty$, $\gamma_{2pn} \to \gamma$, $\gamma_{pn} \to \gamma$, $\gamma_{qn} \to \gamma$

$\gamma + \log 2 + \log p - \frac{1}{2}\gamma - \frac{1}{2}\gamma - \frac{1}{2}\log q - \frac{1}{2}\log p$

$= \log 2 + \log p - \frac{1}{2}\log q - \frac{1}{2}\log p.$

$= \log 2p - \frac{1}{2}\log p\cdot q$

$= \frac{1}{2}\log 4p^2 - \frac{1}{2}\log p\cdot q$

$= \frac{1}{2}\log \frac{4p^2}{pq}$

$= \boxed{\frac{1}{2}\log \frac{4p}{q}}$

Conclusion. for any Real No. α, we can choose (p_n, q_n) [p_n, q_n are arrangement of terms] s.t the Sum of Series can be obtained by Rearranged the series.

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Th^m:- if $\langle a_n \rangle$ is decreasing seq^n of positive Real Number and $\sum a_n$ converges then $\lim_{n\to\infty} n \cdot a_n = 0$.

Result:- if $\sum a_n \cdot x^n$ be a given series then

a) if $\lim_{n\to\infty} \sup |a_n|^{1/n} = 0 \Rightarrow \sum a_n x^n$ cgs Absolutely for every x.

b) if $\lim_{n\to\infty} \sup |a_n|^{1/n} = l \Rightarrow \sum a_n x^n$ cgs Absolutely for $|x| < \frac{1}{l}$ and diverges for $|x| > \frac{1}{l}$.

c) if $\lim_{n\to\infty} \sup |a_n|^{1/n} = \infty \Rightarrow \sum a_n x^n$ cgs only for $x=0$

Result:- $\sum \sin\frac{1}{n}$ series of +ve Real Number is divergent

$\sum \sin n$ Series of Real Number is divergent
$\sum n \sin n$ is divergent

? $\sum \cos n$ –
? $\sum n \cos n$ –

Ex:- $\sum \frac{\sin n}{n^2}$

$$\sum \left| \frac{\sin n}{n^2} \right| \le \frac{1}{n^2}$$

$$\sum \left| \frac{\sin n}{n^2} \right| \le \sum \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ is cgt.

$\Rightarrow \sum \left|\frac{\sin n}{n^2}\right|$ is cgt.

$\Rightarrow \sum \frac{\sin n}{n^2}$ is cgt.

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Qu $\sum \frac{\sin n\pi}{n}$

# $\lim \sin n$ does Not exist But Bdd and more then one limit point is finite oscillate.

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Grouping of terms of Series :- Let $\sum u_n$ be series of +ve Real Number, a New Series $\sum v_n$ from $\sum u_n$ by Grouping the terms of Series and order of terms of series does Not effected. then New series $\sum v_n$ is grouping Series.

Let $\sum u_n$ be Series of +ve Real No. and $\sum v_n$ be grouping Series.

i) if $\sum u_n$ is cgt $\Rightarrow$ $\sum v_n$ is cgt.
ii) if $\sum v_n$ is cgt $\Rightarrow$ $\sum u_n$ is cgt.

Rearranged Series :- $\sum u_n$ be series of Real Numbers $\sum v_n$ be New series from $\sum u_n$ by Rearranged of the terms of the series and order of terms of series is effected. and terms of series exactly one. then New Series $\sum v_n$ is Rearranged Series on u_n.

Let $\sum u_n$ Series of Real No. and $\sum v_n$ be grouping series from $\sum u_n$.

i) if $\sum u_n$ is cgt $\Rightarrow$ $\sum v_n$ is cgt
ii) if $\sum v_n$ is cgt $\not\Rightarrow$ $\sum u_n$ is cgt.

Riemann Theorem :- By appropriate Rearrangement a conditionally cgt series can be made
i) Converge to any Real No.
ii) Diverge to $+\infty$
iii) Diverge to $-\infty$

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iv) oscillate finitely
v) oscillate infinitely

Ex:- $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots = \log 2$.

Result $\langle r^n \rangle$

if $0 < r < 1$ $|r| < 1$

$\lim r^n = 0$

i) if $r = 1$, $\lim r^n = 1$

ii) if $r > 1$; $\lim r^n = \infty$ diverge to $+\infty$

iii) if $r = -1$; oscillate finite

if $r < -1$; oscillate infinite

$r \le -1$

$\lim \left(\frac{-1}{2}\right)^n = \lim \frac{(-1)^n}{2^n} = 0$

If $-1 < r < 0$ then $\lim r^n = 0$

i.e

r^n cgs iff $-1 < r \le 1$

Result $\boxed{\lim r^{1/n} = 1}$, $r > 0$.

Result
$$\lim \frac{a^n}{n^p};\ p > 0 = \begin{cases} 0 & 0 \le a \le 1 \\ 0 & -1 < a < 0 \\ \infty & a > 1 \\ \text{Never converges} & a < -1 \\ \text{(diverge} & \end{cases}$$

Questions.

CSIR - D - 2008.

Qus which of the following Series is absolutely convergen

a) $\sum \frac{(-1)^n}{n}$ b) $\sum \frac{1}{\sqrt{n}}$ c) $\sum \frac{1}{\log(n+1)}$

✓d) $\sum \frac{(-1)^n}{n^{3/2}}$

a) $\sum \frac{(-1)^n}{n}$

$a_n = \frac{1}{n}$

$a_n < a_{n+1} \Rightarrow \downarrow$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

So by Libnitz test $\sum \frac{(-1)^n}{n}$ is cgt

But $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is divergent.

$\Rightarrow$ (a) is conditionaly cgt.

b) $\sum \frac{1}{\sqrt{n}}$ by p test (b) is divergent.

c) $\sum \frac{1}{\log(n+1)}$

$f(n) = \frac{1}{\log(n+1)}$

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$$f(2^n) = \frac{1}{\log(2^n+1)}$$

$$\sum 2^n f(2^n) = \sum 2^n \frac{1}{\log(2^n+1)}$$

$$= \sum 2^n \frac{1}{\log 2^n}$$

$$= \sum \frac{2^n}{n \log 2}$$

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$$= \frac{1}{\log 2} \sum \frac{2^n}{n}$$

$$\frac{2^{n+1}}{n+1} \times \frac{n}{2^n} = \frac{2^n \cdot 2}{n\left[1+\frac{1}{n}\right]} \times \frac{n}{2^n} = \frac{2}{\left[1+\frac{1}{n}\right]}$$

$$\lim_{n\to\infty} \left[\frac{2}{\left[1+\frac{1}{n}\right]}\right] = \lim_{n\to\infty} \frac{2}{1+1/n} = 2$$

$2 > 1$

=) series is divergent.

=) $\sum \frac{1}{\log(n+1)}$ is divergent.

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d) $\sum \frac{(-1)^n}{n^{3/2}}$.

$a_n = \frac{1}{n^{3/2}}$ is $\downarrow$.

$\lim_{n\to\infty} \frac{1}{n^{3/2}} = 0$

then By Libnitz test $\sum \frac{(-1)^n}{n^{3/2}}$ is cgt

$\sum \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum \frac{1}{n^{3/2}}$ is cgt by p-test

$\Rightarrow$ (d) is Absolute convergent.

so option (d) is correct.

NET-I-2009

Q4) Let x be in $\mathbb{R}$. Then the series $\sum x^n e^{-n}$ is

a) Always convergent b) converges only at $x=0$

c) converges if $x<e$ d) converges if $|x|<e$.

we know that $\sum a_n x^n$

if $\lim_{n\to\infty} \sup |a_n|^{1/n} = l$

$\sum a_n x^n$ conv. Absolutely for $|x| < \frac{1}{l}$ and diverges for $|x| > \frac{1}{l}$.

$\lim_{n\to\infty} \sup |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{e} = \frac{1}{e}\ (l)$ then

$\sum a_n x^n = \sum x^n e^{-n}$ is cgt for $|x| < l$

i.e $|x| < e$.

So option d) is correct.

CSIR-J-2013

Que which of the following Series is cgt

a) $\sum_{1}^{\infty} \frac{1}{\sqrt{n+1}-\sqrt{n}}$

[✓ b)] $\sum_{1}^{\infty} \frac{\sin n}{n^2}$

c) $\sum_{1}^{\infty} (-1)^n \log n$

d) $\sum_{1}^{\infty} \frac{\log n}{n}$

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Option (a) $\because \sqrt{n+1}-\sqrt{n} < \sqrt{n}$

$$\frac{1}{\sqrt{n+1}-\sqrt{n}} > \frac{1}{\sqrt{n}}$$

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and $\sum \frac{1}{\sqrt{n}}$ is divergent

$\Rightarrow \sum \frac{1}{\sqrt{n+1}-\sqrt{n}}$ is divergent

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option (b) $\frac{\sin n}{n^2} < \frac{1}{n^2}$

$\sum \frac{\sin n}{n^2} < \sum \frac{1}{n^2} \rightarrow$ cgt

$\Rightarrow \sum \frac{\sin n}{n^2}$ is cgt.

option→3 $\sum a_n = (-1)^n \log n$.

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Here $\lim_{n\to\infty} (-1)^n \log n \neq 0$

$\Rightarrow \sum (-1)^n \log n$ is Not cgt.

option (b) :- $\sum \frac{\log n}{n}$

$f(n) = \frac{\log n}{n}$

$f(2^n) = \frac{\log 2^n}{2^n}$

$\sum 2^n f(2^n) = \sum \log 2^n$.

$\sum n \log 2$.

$\log 2 \sum n \to$ divergent.

$\Rightarrow \sum \frac{\log n}{n}$ is divergent.

so option (b) is correct.

Note:- 1) $\sum_{2}^{\infty} \frac{1}{n \log n}$ is div. [By condensation test.

2) $\sum_{2}^{\infty} \frac{1}{n(\log n)^p}$, is cgt if $p > 1$ & div. if $p \leq 1$

3) Note:- $\sum_{2}^{\infty} \frac{1}{(\log n)^p}$ div. for every $p \in \mathbb{R}$.

June-2015

Qus. $a_n = \frac{1}{n\log n}$; $n \geq 2$ then

i) Seqn $\langle a_n \rangle_{n \geq 2}$ is cgt ✓

ii) Series $\sum_{n \geq 2}^{\infty} a_n$ is cgt

iii) Series $\sum_{n=1}^{\infty} a_n^2$ is cgt.

iv) $\sum (-1)^n a_n$ is cgt.

(i) $\lim_{n\to\infty} \frac{1}{n\log n} = 0$ limit point.

bdd + unique limit point

$\Rightarrow \langle a_n \rangle$ is cgt.

So option (a) is correct.

(ii) $\sum \frac{1}{n\log n}$ is divergent $p \leq 1$

(iii) $\sum \frac{1}{n^2(\log n)^2}$ is cgt $p > 1$

(iv) $\sum (-1)^n a_n$ is cgt by Libnitz test.

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Que→ Consider the series $x + \frac{x^2 \cdot 2^2}{2!} + \frac{x^3\, 3^3}{3!} + \cdots$,

The Series is cgt for.

~~a)~~ $x < \frac{1}{e}$ b) $x >$ [illegible]

c) $-\frac{1}{e} < x < \frac{1}{e}$ d) $x > 0$

$$a_n = \frac{x^n n^n}{n!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{x^n\, n^n}.$$

$$= \lim_{n\to\infty} \frac{\cancel{x^n}\cdot x\,(n+1)^n\cdot\cancel{(n+1)}}{\cancel{(n+1)}\,\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{x^n}\cdot n^n}$$

$$\lim_{n\to\infty} \frac{\cancel{n^n}\left[1+\frac{1}{n}\right]^n x}{\cancel{n^n}} = \lim_{n\to\infty} e^{\frac{1}{n}\times n} x = e x$$

i'e $e x < 1 \Rightarrow \sum a_n$ is cgt.

i'e $x < \frac{1}{e}$.

So option (a) is correct.

Que Consider the series $1 - \frac{1}{4\cdot 3} + \frac{1}{4^2\cdot 5} - \frac{1}{4^3\cdot 7} + \cdots$ is

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✓a) Convergent b) Divergent

c) Oscillate finitely d) None of these.

Que Select the correct Statements

×a) Finite set has infinite Number of limit point.

×b) The seqn $1+r+r^2+\dots+r^n$ where $-1 \le r < 1$ and $n \in \mathbb{N}$ is bdd above but Not bdd below.

✓c) if $u_n > 0 \ \forall n$ then $\sum u_n$ and $\sum \frac{1}{u_n}$ cgs or diverges together.

✓d) Set of Real Number except the integers which are multiples of 9 is uncble set.

G.S. $|x|<1$ cgs ⇒ bdd. so (b) is incorrect.

3Que - $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{3n+k}$ is

a) $\log \frac{4}{3}$ b) $\log \frac{3}{4}$ $\log \frac{3}{2}$

d) $\log \frac{5}{4}$.

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Qu) The series $\sum\left(\sin\frac{1}{n}+\frac{1}{n^2}\right)$ is

a) Divergent b) conditional cgt
c) cgt d) Absolutely cgt

Since $\sum \sin\frac{1}{n}$ is divergent and $\sum\frac{1}{n^2}$ is cgt

and we know that $\sum u_n$ is cgt and $\sum v_n$ is divergent then $\sum u_n \pm v_n$ is divergent.

So option (a) is correct.

Qu) $\sum\frac{n^n}{(n+a)^n}\cdot a$; $a>0$

a) converges for all $a>1$
b) converges for $a>0$
c) converges for $a\geq 0$
d) Divergent for all a.

$$a_n = \frac{n^n}{(n+a)^n}\cdot a \qquad a>0.$$

$$a_n = \left(\frac{n}{n+a}\right)^n\cdot a$$

By Root test $\lim\limits_{n\to\infty}(a_n)^{1/n} = a < 1 \Rightarrow$ cgt

$a>1$ divergent

$a=1$ test fail. Now check at $a=1$

$a_n = \frac{n^n}{(n+1)^n} \cdot 1$ ...

$\lim_{n\to\infty} a_n = 1 \neq 0 \Rightarrow$ Not convergent

$\Rightarrow \sum \frac{n^n}{(n+a)^n} \cdot a$ is Divergent for all a
i.e $a > 1$.

So option (d) is correct.

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Que The Series $\frac{1}{2\cdot3} + \frac{2}{3\cdot4} + \frac{3}{4\cdot5} + - - -$ to ∞, is

a) Convergent [✓ b) Divergent]
c) osi'c oscillating d) None of these

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$a_n = \frac{\cancel{n+1}}{\cancel{n+}}$

$a_n = \frac{n}{(n+1)(n+2)} = \frac{n}{n^2+3n+2}$

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a_n is seq^n

$\lim_{n\to\infty} n\, a_n = \lim_{n\to\infty} n \times \frac{n}{n^2+3n+2}$

$= \lim_{n\to\infty} \frac{n^2}{n^2\left[1+\frac{3}{n}+\frac{2}{n^2}\right]} = 1 \neq 0$

$\Rightarrow$ Not cgt.

Qu) The series $\frac{x}{6}+\frac{5x^2}{10}+\frac{13x^3}{18}+\dots+\frac{2^{n+1}-3}{2^{n+1}+1}x^n+$

is cgt for.

a) $0<x<1$ b) $0<x<$
c) $0<x<3$ d) $x>$

Qu) The series $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\dots$ to ∞ is cgt fo

✓a) $p>0$ b) $p>\frac{1}{2}$ c) $p>1$ d) $p>2$

Qu) Choose the correct statements.

a) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^{\log n}}$ is divergent.

✓b) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^{\log n}}$ is cgt

c) The series $\frac{1}{5}+\frac{2!}{5^2}+\frac{3!}{5^3}+\frac{4!}{5^4}+\dots$ is cgt.

✓d) The series $\frac{1}{5}+\frac{2!}{5^2}+\frac{3!}{5^3}+\frac{4!}{5^4}+\dots$ is divergent

b) by Cauchy condensation test is cgt.

d) $\frac{n!}{5^n}$ $a_n=\frac{n!}{5^n}$

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ Not cgt.

Que :- Consider the series $\sum a_n = 1 + \frac{1!\,x}{2} + \frac{2!\,x^2}{3^2} +$

$(x>0)$ then correct statements are is

a> $\sum a_n$ is cgt for $0<x<1$

[✓ b>] $\sum a_n$ is divergent for $0<x<1$

c> $\sum a_n$ is cgt for $0<x<e$ and diverges for $x>e$

d> $\sum a_n$ is converges for $0<x<e$ and diverges for $x>e$.

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② Que :- choose the Correct statement

a) if $\sum a_n$ is convergent then $\sum (2)^n a_{(2)^n}$ is also cgt provided $a_n \geq a_{n+1} \geq 0 \ \forall \ n \in \mathbb{N}$

b) if $\sum a_n$ is divergent then $\sum (2)^n a_{(2)^n}$ is also divergent provided $a_n \geq a_{n+1} \geq 0 \ \forall \ n \in \mathbb{N}$

[✓ c)] if $\sum (2)^n a_{2^n}$ is cgt then $\sum a_n$ is also cgt provided $a_n \geq a_{n+1} \geq 0 \ \forall \ n \in \mathbb{N}$

[✓ d)] if $\sum (2)^n \cdot a_{2^n}$ is divergent $a_n \geq a_{n+1} \geq 0 \ \forall \ n \geq \mathbb{N}$.

Que if $b_n = \begin{cases} 1/\sqrt{n} & \text{if } n \text{ is odd} \\ 1/n & \text{if } n \text{ is even} \end{cases}$; then

a) both $\langle b_n \rangle$ and $\sum b_n$ are cgt.
b) both $\langle b_n \rangle$ and $\sum b_n$ are divergent
[✓ c)] $\langle b_n \rangle$ is cgt but $\sum b_n$ Not
d) $\langle b_n \rangle$ is Not cgt but $\sum b_n$ is cgt

Que ? Choose the correct answer, where $a_n \geq 0$

[✓ a)] if $\sum a_n$ is cgt then $\sum a_n^2$ is also cgt.
b) if $\sum a_n$ is cgt then $\sum \sqrt{a_n}$ is also cgt
[✓ c)] if $\sum a_n$ is cgt then $\sum \frac{a_n}{a_{n+1}}$ also cgt
d) if $\sum a_n$ is cgt then $\sum \frac{a_n}{1-a_n}$ (provided $a_n \neq 1$) is also cgt.

(i) $\sum \frac{1}{n^2}$ is cgt

$\sum \frac{1}{n^4}$ is also cgt

$\Rightarrow$ (a) is correct.

ⓑ $\sum \frac{1}{n^2}$ is cgt

$\sum \sqrt{\frac{1}{n^2}} = \sum \frac{1}{n}$ is divergent

(b) is incorrect

ⓒ $\frac{a_n}{a_{n+1}} < a_n$

$\sum \frac{a_n}{a_{n+1}} < \sum a_n \to cgt \Rightarrow \sum \frac{a_n}{a_{n+1}}$ is cgt so true

Que: $\sum a_n = a+b+a^2+b^2+a^3+b^3+ \cdots$, then

a) $\sum a_n$ diverges $\forall$ a, b
✓ b) $\sum a_n$ is cgs iff $|a|<1$, $|b|<1$
c) $\sum a_n$ is cgs iff $|a|<1$, $|b|>1$
d) $\sum a_n$ is cgs iff $|a|>1$, $|b|<1$

$a+b+a^2+b^2+a^3+b^3 \cdots$
$= (a+a^2+a^3+ \cdots) + (b+b^2+b^3+ \cdots)$
$\sum a_n + \sum b_n$

$\sum a_n$ & $\sum b_n$ are G.P with common Ration a & b Respectively.

then $\sum a_n$ is cgt $|a|<1$ and $\sum b_n$ is cgt if $|b|<1$

Que:- The value of $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{n}{k^2+n^2}$ equals.

✓ a) 1 b) $\frac{\pi}{4}$ c) π d) 2

Que:- If $\langle a_n \rangle$ is bdd decreasing seq^n of positive Real No. and $\sum a_n$ is cgt. then

a) $\lim\sup a_n = \sup \langle a_n \rangle$
b) $\lim\inf a_n = \inf \langle a_n \rangle \neq 0$
✓ c) $\lim\inf a_n = \lim\sup a_n =$ sum of Series
d) $\lim\sup a_n = \lim\inf a_n = 0$

Qu) The series $\frac{1}{2^p}+\frac{1}{2^p+1}-\frac{2}{2^p+2}+\frac{1}{3^p}+\frac{1}{3^p+1}\ \frac{2}{3^p+2}+\frac{1}{4^p}$
$+\frac{1}{4^p+1}\ \frac{2}{4^p+2}+ \cdots$

a) converges if $p \in [1,\infty)$ and diverges if $p \in (-\infty,1)$
b) converges if $p \in (0,\frac{1}{2}]$ and diverges if $p \in (\frac{1}{2},\infty)$
c) converges if $p \in (\frac{1}{2},\infty)$ and diverges if $p \in (0,\frac{1}{2}]$
d) None of these.

Qu) Match list I and list II

List I

A. $\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{n^2}$

B. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{n}\right)$

? C. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

1 D. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

E. $\sum\left(\sin\frac{1}{n} + \frac{1}{n^2}\right)$

List II

1) Oscillates finitely
2) Convergent
3) Conditionally conv.
4) Absolute Convergent
5) Divergent

code	A	B	C	D	E
(a)	1	2	4	5	3
(b)	3	2	4	5	1
[✓ (c)]	3	1	2	4	5
(d)	5	1	2	4	3

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Que The series $\sum_{k=1}^{\infty}\left(\frac{x}{1+x} \sin nx\right)^{k}$ on $[0,1]$

a) converges pointwise
b) converges uniformly
c) Does not converge point wise
d) converges pointwise to a discountinous function

Que. The Series $\sum \frac{1}{4}\{1+(-1)^{n+1}(2n+1)\}$ is

a) convergent
[✓ b) Does Not converge]
c) Diverge
d) oscillates infinitely

Que :- The series $\sum_{n=1}^{\infty} e^{\sqrt{n}} x^{n}$ is convergent for

[✓ a) $0 < x < 1$] b) $x > 1$
c) $0 < x < 2$ d) $x > 2$

$a_n = e^{\sqrt{n}}$.

$\lim\limits_{n\to\infty} \sup (a_n)^{1/n} = \left(e^{\sqrt{n}}\right)^{1/n} = e^{1/\sqrt{n}}$

$\lim\limits_{n\to\infty} e^{1/\sqrt{n}} = e^{1/\infty} = e^{0} = 1$

i.e $|x| < 1$ is cgt

$-1 < x < 1$ is cgt

Qus→ The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \csc^{-1} n$ is.

a) Convergent b) divergent

c) oscillating d) N.O.T

Qus If $x_n = \sum_{t=0}^{n-1} r^t$ with $|r| < 1$ then the seq^n $\langle x_n \rangle$ converges to

a) 0 b) 1 c) $\frac{1}{1-r}$ d) $\frac{1}{1+r}$

Qus Which of the following Series is / are Not cgt.

(a) $\sum_{1}^{\infty} \sqrt{n} \sin\frac{1}{n}$ b) $\sum_{1}^{\infty} \frac{1}{n} \cot^{-1} n$

c) $\sum_{1}^{\infty} \csc^{-1} n$ d) $\sum_{1} \tan\frac{1}{n}$

Que if Σa_n & Σb_n are cgt and divergent series of Real No. Respectively. Then

a) $\langle a_n \rangle$ is Monotonic
b) $\langle b_n \rangle$ is divergent
c) Seqn of partial sums of Σb_n is divergent
~~d) $\lim a_n = 0$~~

Que - Let $\langle a_n \rangle$ & $\langle b_n \rangle$ two seqn of Real Numbers s.t $a_n = b_n - b_{n+1}$ $\forall$ $n \in \mathbb{N}$. Then

a) if Σa_n converges $\Rightarrow$ $\langle b_n \rangle$ converges but Not converse
b) if Σa_n converges $\Rightarrow$ Σb_n converges but Not conversely
~~c) Σa_n converges $\Leftrightarrow$ $\langle b_n \rangle$ converges~~
~~d) Σb_n converges $\Leftrightarrow$ $\langle a_n \rangle$ converges.~~

$$a_n = b_n - b_{n+1}$$

$$S_n = b_1 - b_2 + b_2 - b_3 - - - - + b_n - b_{n+1}$$

$$S_n = b_1 - b_{n+1}$$

$$\lim_{n\to\infty} S_n = \text{finite. iff } \langle b_n \rangle \text{ is cgt.}$$

i.e Σa_n converges iff $\langle b_n \rangle$ is cgs.

Que - The series $1 - \frac{1}{2^3}(1+2) + \frac{1}{3^3}(1+2+3) - - - - -$ to ∞ is

a) Conditionally convergent
b) Divergent
c) Absolute convergent
d) Oscillating

$a_n = \sum (-1)^n \frac{n(n+1)}{2n^3}$

M.I. $\lim_{n\to\infty} \frac{n(n+1)}{2n^3} = 0 \Rightarrow$ series is cgt But

Not Absolute cgt.

Ques:- Which of the following series is / are Not cgt.

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n+1}$ b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^2+1}$

c) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{n^3+3}$ d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^2+1}$

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$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n+1}$

$u_n = \frac{\sqrt{n+1}}{n+1} \sim \frac{\sqrt{n}}{n} \sim \frac{1}{\sqrt{n}} = v_n$

$\frac{u_n}{v_n} = \frac{\sqrt{n+1}}{n+1} \times \frac{n^{1/2}}{1}$

$\lim_{n\to\infty} \frac{\cancel{n}\left[\sqrt{1+\frac{1}{n}}\right]}{\cancel{n}\left[1+\frac{1}{n}\right]} = 1 \neq 0$ finite.

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and v_n is divergent.

$\Rightarrow$ u_n is divergent i.e Not cgt.

Similarly (b) and (c) are divergent.

Que - If $\sum u_n$ is a Series of positive term then

a) $\sum (-1)^n u_n$ cgt $\Rightarrow$ $\sum u_n$ converges

b) $\sum u_n$ converges $\Rightarrow$ $\sum (-1)^n u_n$ converges

c) $\sum u_n$ converges $\Rightarrow$ $\sum u_n^2$ converges.

d) $\sum u_n^2$ converges $\Rightarrow$ $\sum u_n$ converges.

Soln - Ex $\sum \frac{(-1)^n}{n}$ is cgt But Not Ab. cgt.

i.e $\sum \frac{1}{n}$ is Not cgt So option (a) is incorrect

? Que - Let $\sum a_n = \sum \frac{n!}{x(x+1)(x+2)\cdots(x+n-1)}$ then

a) $\sum a_n$ converges if $x>2$

b) $\sum a_n$ is divergent if $x>2$

c) $\sum a_n$ diverges if $x \le 2$

d) $\exists$ $x_0 \in \mathbb{R}$ s.t. $\sum a_n$ is oscillates

NET·J-2012

Qn Using the fact that $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ then

$\sum_{1}^{\infty} \frac{1}{(2n+1)^2}$ equal

a) $\frac{\pi^2}{12}$ b) $\frac{\pi^2}{12} - 1$ c) $\frac{\pi^2}{8}$ d) $\frac{\pi^2}{8} - 1$

Soln:- $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

$\sum \frac{1}{(2n+1)^2} = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + - - -$

$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + - - - - \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + - -\right] - 1$

$= \frac{\pi^2}{6} - \frac{1}{4}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + - - -\right] - 1$

$= \left[\frac{\pi^2}{6} - \frac{\pi^2}{24}\right] - 1$

$\frac{4\pi^2 - \pi^2}{24} = \frac{3\pi^2}{24} - 1 = \frac{\pi^2}{8} - 1$

$= \frac{\pi^2}{8} - 1$

i.e option (d) is correct.

2 Qn Consider the Series $\sum \frac{n!}{3^n}$ which of the following statement are true.

a) Series convergent to a value $> 7/4$

b) Series convergent to a value < 100

c) Series convergent to a value < 10

d) Series divergent

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(NET J-2011)

Que Using the fact $\sum \frac{(-1)^{n+1}}{n} = \log 2$, $\sum_{1}^{\infty} \frac{(-1)^n}{n(n+1)}$ equals

[✓ a)] $1-2\log 2$ b) $1+\log 2$

c) $(\log 2)^2$ d) $-(\log 2)^2$

Sol^n - $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - - = \log 2$

$$\sum_{1}^{\infty} \frac{(-1)^n}{n(n+1)} = \sum_{1}^{\infty} (-1)^n \left[\frac{1+n-n}{n(n+1)}\right] = \sum_{1}^{\infty} (-1)^n \left[\frac{1}{n} - \frac{1}{(n+1)}\right]$$

$$= \sum (-1)^n \left[\frac{1}{n} - \frac{1}{(n+1)}\right]$$

$$= \sum \frac{(-1)^n}{n} - \sum \frac{(-1)^n}{n+1}$$

$$= -\log 2 - [\log 2 - 1]$$

$$= -\log 2 - \log 2 + 1$$

$$= 1 - 2\log 2.$$ Ans

Que Let $\sum_{n=1}^{\infty} a_n$ is cgt but Not absolutely cgt and $\sum_{n=1}^{\infty} a_n = 0$. Denote S_k the Partial sum $\sum_{n=1}^{k} a_n$ & $k = 1, 2, 1$ —— then

a) $S_k = 0$ for infinitely many k

b) $S_k > 0$ for infinitely many k and $S_k < 0$ for infinitely many k

c) It is possible ~~that~~ that $S_k > 0$ for all k

d) It is possible that $S_k > 0$ for all but finite Number of values of k.

Que Let $\langle a_n \rangle$ be the sequence given by

$$a_n = \int_{-\infty}^{\infty} \frac{\cos(nx)}{1+x^2}\, dx, \text{ then}$$

a) $\langle a_n \rangle$ is bdd

b) $\langle a_n \rangle$ is bdd doesn't converge

c) $\langle a_n \rangle$ converges but $\sum_{-\infty}^{\infty} a_n$ diverges

d) $\sum_{n=-\infty}^{\infty} a_n$ converges

Que $\langle x_n \rangle$ be a seq^n of complex numbers, then we must have

a) $\sum_{1}^{\infty} x_n$ converges   b) $\sum_{1}^{\infty} x_n^2$ converges

c) $\exists$ subseq^n $\langle x_{n_k} \rangle$ s.t. $\sum_{1}^{\infty} 2^k x_{n_k}$ converges

d) $\nexists$ subseqn $\langle x_{n_k} \rangle$ s.t. $\sum_{1}^{\infty} 4^k x_{n_k}$ converges.

Que - The Sum of Series $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \text{---} + \frac{1}{100\cdot101}$ is

a) $\frac{99}{101}$ b) $\frac{98}{101}$ c) $\frac{99}{100}$ d) N.O.T

Qu The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

a) Converges but Not absolutely
b) Converges absolutely
c) diverges
d) N.O.T

Qu Let $u_n = \frac{\sin n\pi}{n}$ and consider the series $\sum u_n$ then False statements

a) $\sum u_n$ is Convergent
b) $u_n \to 0$ as $n \to \infty$
c) $\sum u_n$ is divergent
d) $\sum u_n$ is absolutely convergent.

Qu $\langle a_n \rangle$ be a sequence of positive terms, pickout cases which imply that $\sum a_n$ is convergent:

a) $\lim_{n\to\infty} n^{3/2} a_n = \frac{3}{2}$

b) $\sum n^2 a_n^2 < \infty$

c) $\frac{a_{n+1}}{a_n} < \left(\frac{n}{n+1}\right)^2 \ \forall \ n \in \mathbb{N}$

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Q-> Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{4n^2-k^2}} = 2$

Soln $\sum \frac{1}{n} \frac{n}{\sqrt{4n^2-k^2}} = a_n = \frac{n}{\sqrt{4n^2-k^2}} = \frac{n}{n\sqrt{4-\frac{k^2}{n^2}}}$

$\lim_{n\to\infty} \frac{n}{n\sqrt{4-k^2/n^2}} = \frac{1}{2}.$

Que:- Pick out the convergent series.

a) $\sum_{1}^{\infty} \frac{(n+1)^n}{n^{n+3/2}}$? b) $\sum_{1}^{\infty} \frac{\sqrt{1-4^n}}{1+5^n}$

c) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+ - - -$

$a_n = \sum_{1}^{\infty} \frac{(n+1)^n}{n^{n+3/2}}.$

$u_n = \frac{(n+1)^n}{n^{n+3/2}} \sim \frac{n^n}{n^{n+3/2}} = \frac{1}{n^{3/2}} = v_n$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0$

$\Rightarrow \sum v_n$ is cgt $\Rightarrow \sum u_n$ is cgt.

c) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^2}{4^4}+ - -$

a_n is M↓.

$\lim_{n\to\infty} n u_n = \frac{n(n)^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{(n)^{\cancel{n+1}}}{\cancel{n^{n+1}}\left[1+\frac{1}{n}\right]^{n+1}} = 1 \neq 0$

⇒ divergent.

Q→44 In each of the Following cases, state whether the Series is absolutely conv, conditionally conv. or divergent.

a) $\sum (-1)^n \frac{1}{2n+3}$ b) $(-1)^n \frac{n}{2n+2}$ c) $\sum (-1)^n \frac{n \log n}{e^n}$

$\sum (-1)^n \frac{1}{2n+3}$ is conditionally cgt.

$\sum (-1)^n \frac{n}{n+2}$ is cgt.

Que:- Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \cos\left(\frac{\pi k}{2n}\right)$

$a_n = \cos \frac{\pi k}{2n}$, $\lim_{n\to\infty} \cos \frac{\pi n}{2n} = 0$

$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \cos\left(\frac{\pi k}{2n}\right) = 0$ Ans.

Que Pick out the Series which are absolutely cgt.

a) $\sum_{1}^{\infty} (-1)^n \frac{\cos n\alpha}{n^2}$ $\alpha \in \mathbb{R}$ is a fixed Real Number

b) $\sum_{1}^{\infty} (-1)^n \frac{\log n}{e^n}$ (c) $\sum (-1)^n \frac{n}{n+2}$

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Que $\lim_{n\to\infty} \sum_{1}^{\infty} \frac{1}{k(k+1)}$ is equal to

a) 0 b) 1 c) $\log_e 2$ d) ∞

$a_n = \frac{1}{n(n+1)} \sim \frac{1}{n^2} = v_n$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0$

$\Rightarrow \sum \frac{1}{k(k+1)}$ is cgt and $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 0$

Que $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{k}{n^2}$, calculate the lim of given series.

$a_n = \frac{n}{n^2}$

$\frac{a_{n+1}}{a_n} = \frac{n+1}{(n+1)^2} \times \frac{(n)^2}{n} = \frac{\cancel{n}(1+1/n) \times \cancel{n^2}}{\cancel{n^2}(1+1/n)^2 \times \cancel{n}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ i.e. $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{k}{n^2} = 1$ & Not cgs.

Que $\sum a_n$ is cgt & $\sum b_n$ is divergent series of positive term, then

a) $\langle a_n \rangle$ is convergent and $\langle b_n \rangle$ is Not cgt

b) $\langle a_n \rangle$ converges to 0

c) $\langle b_n \rangle$ doesn't converge to 0

d) $\langle b_n \rangle$ diverges to ∞

Que Which of the following Series is absolutely convergent?

a) $\sum \frac{(-1)^n}{n}$ b) $\sum \frac{1}{\sqrt{n}}$ c) $\sum \frac{1}{\log(n+1)}$

d) $\sum \frac{(-1)^n}{n^{3/2}}$

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Que The Series $\sum_{n=1}^{\infty} \frac{(2n+1)}{(n^2+n)^2}$

a) converges to 1
b) converges to a Number > 1
c) diverge to $+\infty$
c) has an oscillating seqⁿ of partial sum.

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Que If $\sum a_n^2$ is convergent then

a) $\sum a_n$ is cgt
b) $\sum a_n$ cgt but $\sum \frac{a_n}{n}$ is Not cgt
c) $\sum a_n$ is Not cgt but $\sum \frac{a_n}{n}$ is cgt
d) $\left(\sum a_n^2\right) \le \left(\sum a_n^2\right)^{1/2} . \sum (a_n^2)^{1/2}$

24 (i) Let $x \in \mathbb{R}$, then the Series $\sum_{1}^{\infty} x^{n} e^{-n}$ is

a) Always convergent b) converges only at $x=0$.

b) converges if $x<e$ ~~d)~~ Converges if $|x|<e$

$\lim\sup |a_n|^{1/n} = \lim_{n\to\infty}\sup \left|\frac{1}{e^{n}}\right|^{1/n} = \frac{1}{e}$.

cg if $|x| < \frac{1}{l}$ i.e. $|x| < 1/_{1/e}$ i.e. $|x| < e$.

(ii) $\sum a_n = \sum \frac{(-1)^n \cdot \log n}{n - \log n^{1.01}}$ and

$\sum b_n = \sum \frac{(-1)^n \sin \pi/n}{n}$ then

a) $\sum a_n$ & $\sum b_n$ are absolutely convergent

b) $\sum a_n$ is absolutely convergent but $\sum b_n$ Not

c) $\sum b_n$ is absolutely convergent but $\sum a_n$ is Not.

d) Neither $\sum a_n$ nor $\sum b_n$ is absolutely convergent

$\sum b_n = \sum \left|\frac{(-1)^n \sin \pi/n}{n}\right| = \sum \frac{\sin \pi/n}{n}$

$u_n = \frac{\sin \pi/n}{n} \sim \frac{\pi}{n^2} = v_n$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin \frac{\pi}{n}}{n} \Big/ \frac{\pi}{n^2} = \frac{\sin \frac{\pi}{n}}{\cancel{n}} \times \frac{n^{\cancel{2}}}{\pi}$

$\lim_{n\to\infty} \frac{n \sin \pi/n}{\pi} = \lim_{n\to\infty} \frac{\sin \pi/n}{\pi/n} = 1 \neq 0$

and $\sum v_n$ is cgt ⇒ $\sum u_n$ is cgt.

⇒ $\sum b_n$ is Absolutely cgt.

PART-II

SKM ACADEMY

REAL ANALYSIS

NET/GATE/JAM

LIMIT AND CONTINUITY OF REAL VALUED FUNCTIONS

Limit point :- Let $A \subseteq \mathbb{R}$ a point $c \in \mathbb{R}$ is called limit point of set A if every $\delta > 0$, $\exists$ a $x \in A$, $x \neq c$ s.t $|x-c| < \delta$.

$\boxed{\text{Or}}$

a point $c \in \mathbb{R}$ is called limit point of set $A \subseteq \mathbb{R}$, $\exists$ a seq^n $\langle x_n \rangle$ of A and $x_n \neq c$ $\forall$ $n \in \mathbb{N}$ (or other than c) and $x_n \Longrightarrow c$

Limit of function :- Let $A \subseteq \mathbb{R}$, c is limit point of set A (may not member of A) and $c \in \mathbb{R}$, a function $f : A \longrightarrow \mathbb{R}$ has limit L at c if for every $\epsilon > 0$ $\exists$ a $\delta > 0$ and δ depends on ϵ s.t $|f(x) - L| < \epsilon$ whenever $|x-c| < \delta$ and $x \neq c$

$$\left\{\begin{array}{l} |x-c| < \delta \\ -\delta < x-c < \delta \\ c-\delta < x < c+\delta \\ x \in (c-\delta, c+\delta) \\ A = (c-\delta, c+\delta) = \{x : |x-c| < \delta\} \end{array}\right.$$

or $|f(x) - L| < \epsilon$

$f(x) \in (L-\epsilon, L+\epsilon)$

$= \{f(x) : |f(x) - L| < \epsilon\}$

ie $|f(x) - L| < \epsilon$ whenever $x \in (c-\delta, c+\delta) \cap A - \{c\}$.

We denote this by $\lim\limits_{x \to c} f(x) = L$

$\boxed{\text{Or}}$

$f(x) \longrightarrow L$ as $x \Longrightarrow c$

$|f(x) - L| < \epsilon$ whenever $0 < |x-c| < \delta$

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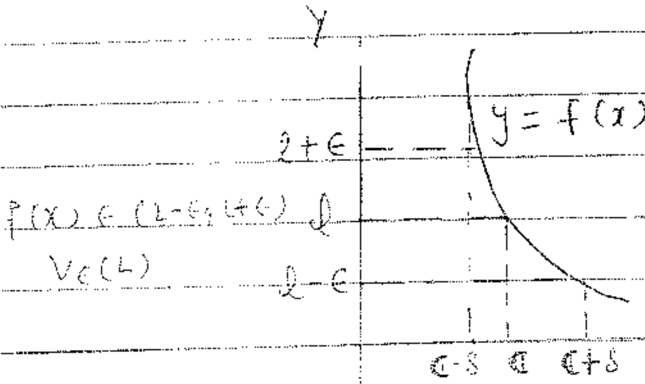
# Let $A \subseteq \mathbb{R}$, c is limits point of set A, a function $f : A \to \mathbb{R}$
then following conditions are equivalent.

~~1) $\lim_{x \to c} f(x) = L$~~

2) $\exists$ a $\delta > 0$, a Nbd of c i.e $V_\delta(c)$ for Every $\epsilon > 0$, a L Nbd i.e $V_\epsilon(L)$, $x \in A$ and $x \neq c$ then $f(x) \in (L-\epsilon, L+\epsilon)$

i.e $f(x) \in (L-\epsilon, L+\epsilon)$ whenever $x \in (c-\delta, c+\delta)$


Y
$y = f(x)$
$l+\epsilon$
$f(x) \in (L-\epsilon, L+\epsilon)$ l
$V_\epsilon(L)$
$l-\epsilon$
$c-\delta$
c
$c+\delta$
X

$V_\delta(c) = (c-\delta, c+\delta)$

Note:- The idea of function having a limit 'l' at the point ~~c~~ $x = c$ is that the value $f(x)$ are close to 'l' when x close to 'c' i.e when x approaches close to 'c', $f(x)$ approaches close to 'l'

Note:- From the defination of limit of function it is cleared that c is limit point of A

may or may Not member of A, then the value of f at c may Not be defined. But it should be defined at enougth points close to 'c'.

Notes:- if l is the limit ~~poin~~ of 'f' at $x=c$ i.e $|f(x)-l| < \epsilon$ whenever $x \in (a-\delta, c+\delta) \cap A - \{c\}$ i.e at 'c' it may fail that $|f(x)-l| < \epsilon$ for $(a-\delta, c+\delta) \cap S$ but always exist in deleated Nbd of c.

Note:- It is Not Necessary that $\lim_{x\to a} f(x) = h$ that $f(a)=h$

E.g:- $f(x) = \begin{cases} 2x+3 & ; x \neq 2 \\ 3 & ; x = 2 \end{cases}$

$\lim_{x\to 2} f(x) = 7$ but $f(2) = 3$

Second Definition [General principal for Existence of limit]-

Let $f : S(\subseteq \mathbb{R}) \to \mathbb{R}$ be a function. Let 'a' be a limit point of S then we say $\lim_{x\to a} f(x)$ exist if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ s.t

$$|f(x_1) - f(x_2)| < \epsilon \ ; \ \forall\ x_1, x_2 \in (a-\delta, a+\delta) \cap S - \{a\}.$$

Que:- Show that $\lim_{x\to 0} \sin\frac{1}{x}$ exist or Not.

Soln: for an $\delta > 0$ we can choose two points x_1, x_2 s.t

$$x_1 = \frac{1}{n\pi/2} < \delta \ , \ x_2 = \frac{1}{(n+1)\frac{\pi}{2}} < \delta$$

Here $x_1, x_2 \in (0-\delta, 0+\delta)$

$|f(x_1) - f(x_2)| = \left|\sin\frac{n\pi}{2} - \sin\frac{(n+1)\pi}{2}\right|$

if $\sin\frac{n\pi}{2} = 1$ then $\sin(n+1)\frac{\pi}{2} = -1$

$\Rightarrow \left|\sin n\frac{\pi}{2} - \sin(n+1)\frac{\pi}{2}\right| = 1$

Conclusion:- if we take $\epsilon = 1/2$ and two points $x_1 < \delta$ & $x_2 < \delta$ s.t. $x_1, x_2 \in (0-\delta, 0+\delta)$ but $|f(x_1) - f(x_2)| = 1 \neq$ or $\nless \epsilon = \frac{1}{2}$, then we can say $\lim_{x\to 0} \sin\frac{1}{x}$ doesn't exist.

(Sequential)

Third Defination ' [Hein's Theorem]

Let $f : S(\subseteq \mathbb{R}) \longrightarrow \mathbb{R}$ be a function and 'a' be a limit point of $\mathbb{R}$. Then we say $\lim_{x\to a} f(x) = l$ iff whenever $\langle a_n\rangle \longrightarrow a$, $\langle f(a_n)\rangle \longrightarrow l$

i.e if 'l' is limit of function at '$x=a$' then $\exists$ a sequence $\langle a_n\rangle$ s.t $\langle f(a_n)\rangle \longrightarrow l$ when $n \longrightarrow \infty$.

Divergence Certoria:-

Note: if there exist a sequence $\langle a_n\rangle$ s.t $\langle a_n\rangle \to a$ as $n\to\infty$ but $\langle f(a_n)\rangle \longrightarrow l_1$ when $n\to\infty$, and one other sequence $\langle b_n\rangle$ s.t $\langle b_n\rangle \longrightarrow a$ as $n\to\infty$ and $\langle f(b_n)\rangle \longrightarrow l_2$ then we say can say Limit doesn't exist at $x=a$.

i.e $\exists$ two seq^n $\langle a_n\rangle$ & $\langle b_n\rangle$ s.t $\langle a_n\rangle \to a$

and $\langle b_n\rangle \to a$ but sequence $\langle f(a_n)\rangle \to l_1$ & $\langle f(b_n)\rangle \to l_2$, then $\lim\limits_{x\to a} f(x)$ doesn't exist.

Eg:- $f(x) = \sin\frac{1}{x}$

let $a_n = \frac{1}{n\pi}$, $\langle a_n\rangle \to 0$

$b_n = \frac{1}{(4n-3)\frac{\pi}{2}}$, $\langle b_n\rangle \to 0$

but $f(a_n) = \sin n\pi = 0$, $\langle f(a_n)\rangle \to 0$

$f(b_n) = \sin (4n-3)\frac{\pi}{2} = 1$, $\langle f(b_n)\rangle \to 1$

$\Rightarrow \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\sin\frac{1}{x}$ doesn't exist.

Note:- if $\lim\limits_{x\to a} f(x)$ exist then the set

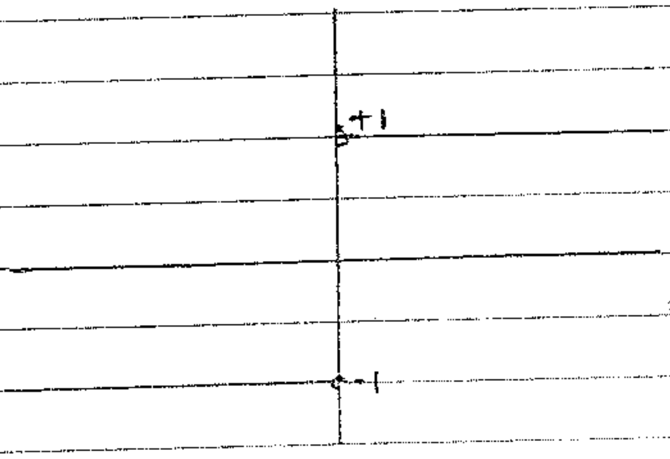
$$S = \left\{ \lim_{n\to\infty} f(a_n) \ ; \ \langle a_n\rangle \longrightarrow a \right\}$$ is singleton.

Pf:- Let $\lim\limits_{n\to\infty} f(x) = l$ then by Hein's Thm whenever seqn $\langle a_n\rangle \longrightarrow a$, sequence $\langle f(a_n)\rangle$ will tend to 'l' as $n\to\infty$ then $S = \{l\}$

OUT

Divergence criteria :- Let $S \subseteq \mathbb{R}$, a is limit point of set S and function $f : S \longrightarrow \mathbb{R}$, function does Not have limit h at a as $x \to a$, $\exists$ a seqn $\langle x_n\rangle$ of S and $x_n \neq a$ $\forall$ $n \in \mathbb{N}$, $x_n \to a$ but $f(x_n) \not\to h$ i.e. $f(x_n)$ does Not tend to h

Eg:- $f(x) = \operatorname{sgn}(x) = \begin{cases} 1 & ; x > 0 \\ -1 & ; x < 0 \\ 0 & ; x = 0 \end{cases} = \begin{cases} \frac{x}{|x|} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$



$\langle x_n \rangle = \frac{(-1)^n}{n}$

$a_n = \frac{1}{n}$ $b_n = \frac{-1}{n}$

$a_n \to 0$ as $n \to \infty$ $b_n \to 0$ as $n \to \infty$.

But $f(a_n) = 1$ and $f(b_n) = -1$

i.e $\langle f(a_n) \rangle \to 1$ as $n \to \infty$

and $\langle f(b_n) \rangle \to -1$ as $n \to \infty$.

$\Rightarrow$ limit Does Not exist at $x = 0$

i.e limit Does Not exist in $\mathbb{R}$.

Bdd:- Let $A \subseteq \mathbb{R}$, c is Limit point of A, $f: A \to \mathbb{R}$ function is said to be bdd in Nbd of c if $\exists$ a +ve M s.t $|f(x)| \leq M$ $\forall$ $x \in A \cap (c-\delta, c+\delta)$

$\boxed{09}$

$\#$ Let $A \subseteq \mathbb{R}$, c is limit point of A, $f: A \to \mathbb{R}$ function has limit L at 'c', then fun. is bdd in Nbd of 'c'

Result #1 if function is Not bdd in Nbd of c then function has No limit l in IR

E.g:- $f(x) = \frac{1}{x}$ at 0

Function is Not bdd in Nbd of zero

$\lim_{x \to 0} f(x)$ = Not Exist at $x = 0$

i.e Limit Does Not Exist in IR

Right hand limit (R.H.L):-

Let 'a' be a limit point of S set $(a, a+\delta) \cap S$ is infinite $\forall\ \delta > 0$, we say 'f' has Right hand limit (R.H.L) if for any $\epsilon > 0$ $\exists\ \delta > 0$ s.t $|f(x_1) - f(x_2)| < \epsilon$ $\forall\ x_1, x_2 \in (a, a+\delta) \cap S$ and denoted by $\lim_{x \to a^+} f(x)$.

Note:- if $\forall$ seq^n $\langle a_n \rangle \longrightarrow a$ where $a_n \in (a, a+\delta) \cap S$ and $\lim_{n \to \infty} f(a_n) = l \Rightarrow \lim_{x \to a^+} f(x) = l$

i.e if $\exists$ a seq^n $\langle a_n \rangle \longrightarrow a$, $a_n \in (a, a+\delta) \cap S$ but $\lim_{n \to \infty} f(a_n) \not\longrightarrow l \Rightarrow \lim_{x \to a^+} f(x)$ doesn't exist

Left hand limit (L.H.L):-

Let 'a' be a limit point of S set $(a-\delta, a) \cap S$ infinite $\forall\ \delta > 0$. we say 'f' has L.H.L if for any $\epsilon > 0$ $\exists\ \delta > 0$ s.t $|f(x_1) - f(x_2)| < \epsilon$ $\forall$ $x_1, x_2 \in (a-\delta, a) \cap S$.

denoted by $\lim_{x\to a^-} f(x)$

Note:- if $\forall \langle a_n\rangle \to a$, $a_n \in (a-\delta, a)\cap S$ & $\lim_{n\to\infty} f(a_n) = l \Rightarrow \lim_{x\to a^-} f(x) = l$.

Note:- Limit exist if R.H.L & L.H.L both exist and equal.

How to find R.H.L and L.H.L:-

First put $x = a+h$ and $x = a-h$ in $f(x)$ for R.H.L and L.H.L Respectively $(h>0)$, then

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0^+} f(a+h) \quad [\text{For R.H.L}]$$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0^-} f(a-h) \quad [\text{For L.H.L}]$$

Eg:- Find L.H.L and R.H.L for $f(x) = 2^{\frac{1}{x-1}}$ when $x \to 1$?

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} \underset{f(1+h)}{f(a+h)} = \lim_{h\to 0} 2^{\frac{1}{1+h-1}} = \lim_{h\to 0} 2^{\frac{1}{h}} = \infty$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0^-} f(a-h) = \lim_{h\to 0^-} f(1-h)$$

$$= \lim_{h\to 0^-} 2^{\frac{1}{1-h-1}} = \lim_{h\to 0^-} 2^{-\frac{1}{h}} = 2^{-\infty} = 0$$

So R.H.L $= \infty$ and L.H.L $= 0$

Result:- Let $A \subseteq \mathbb{R}$ and c is limit point of A, function

? $f : A \to \mathbb{R}$

Both condition are equivalent

i) $\lim\limits_{x \to c^+} f(x) = L$ and $x > c$ $\forall n \in \mathbb{N}$

ii) $\lim\limits_{x \to c^-} f(x) = L$ and $x < c$ $\forall n \in \mathbb{N}$.

Limit at infinity and infinite limits :-

1) $\lim\limits_{x \to \infty} f(x) = l$ ⇒ A function $f(x)$ said to tend to l as $x \to \infty$ if given $\epsilon > 0$ however small, $\exists$ $K \in \mathbb{R}^+$ (depending on ϵ) s.t $|f(x) - l| < \epsilon$ $\forall$ $x \geq K$

i.e $l - \epsilon < f(x) < l + \epsilon$ $\forall$ $x \geq K$.

2) $\lim\limits_{x \to -\infty} f(x) = l$ ⇒ $|f(x) - l| < \epsilon$ $\forall$ $x \leq -K$

i.e $l - \epsilon < f(x) < l + \epsilon$ $\forall$ $x \leq -K$

then $f(x)$ said to tend to l as $x \to \infty$

3) $\lim\limits_{x \to a} f(x) = \infty$ ⇒ A function $f(x)$ is said to tend to ∞ as $x \to a$ if given $K > 0$, However large, $\exists$ a +ve Number K s.t $f(x) > K$ for $0 < |x - a| < \delta$

4) $\lim\limits_{x \to a} f(x) = -\infty$ ⇒ $f(x) < -K$ for $0 < |x - a| < \delta$, then $f(x)$ said to tend to $-\infty$ as $x \to a$

5) $\lim\limits_{x \to \infty} f(x) = \infty$ ⇒ $f(x)$ said to tend to ∞ as $x \to \infty$ if given $K > 0$ However large, $\exists$ $K' > 0$ s.t $f(x) > K$ $\forall$ $x \geq K'$

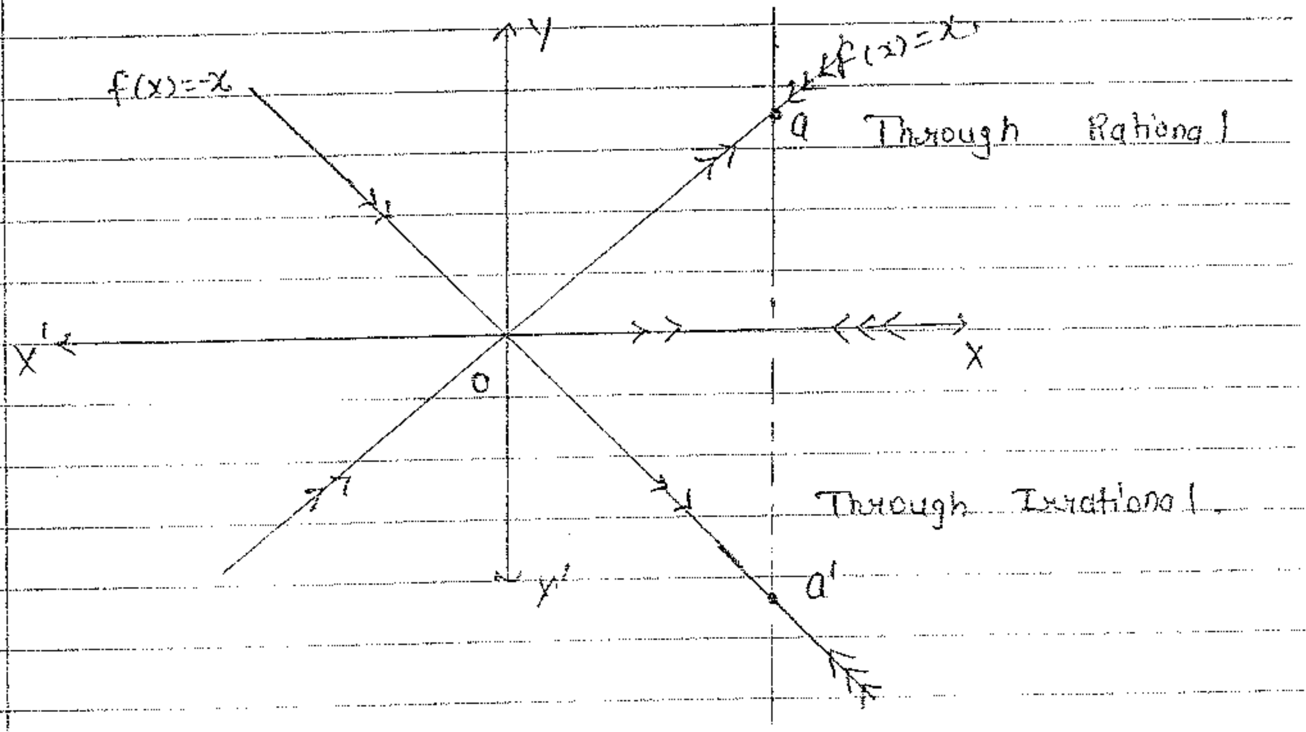
6) $\lim_{x\to\infty} f(x) = -\infty \Rightarrow f(x) < -K \;\forall\; x \geq K'$ then $f(x)$ said to tend to $-\infty$ as $x\to\infty$

7) $\lim_{x\to -\infty} f(x) = \infty \Rightarrow f(x) > K \;\forall\; x \leq -K'$

8) $\lim_{x\to -\infty} f(x) = -\infty \Rightarrow f(x) < -K \;\forall\; x \leq -K'$

Qus :- What is the behaviour of $f(x)$ when x approches to 1, function given as -

$$f(x) = \begin{cases} x & ; \; x \in \mathbb{Q} \\ -x & ; \; x \in \mathbb{Q}^c \end{cases}$$



f(x)=-x
Y
f(x)=x
a
Through Rational
X'
X
O
Through Irrational
a'
Y'

$\lim_{x\to 1^+} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = -1$

$\Rightarrow$ Right hand limit doesn't exist at $x=1$

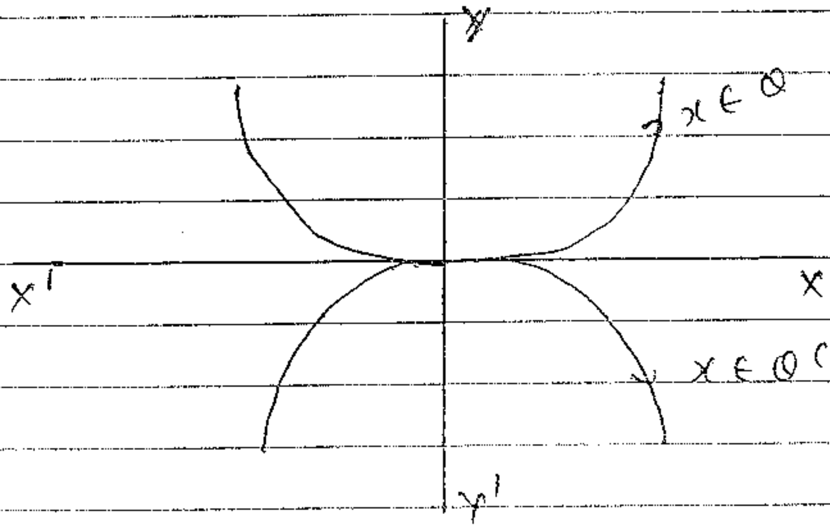
Similarly $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^-} f(x) = -1$

i.e. left hand limit doesn't exist at $x=1$.

Hence in all $\lim_{x \to 1} f(x)$ doesn't exist.

only $x = 0$ is the point where limit exist and equal to 0 (by seeing the graph, Right and Left limit both ~~e~~ will exist and will be equal and equal to 0

Que $f(x) = \begin{cases} x^2 & ; x \in \mathbb{Q} \\ -x^2 & ; x \in \mathbb{Q}^c \end{cases}$



Discuss the limit at $x = a$ where a is any point of $\mathbb{R}$ [same as previous]

Result :- if f is Monotonic function define on (a, b) and $c \in (a, b)$ then at 'c' R.H.L and L.H.L both exist but may or may not be equal.

Thm :- The limit of a function at a point is unique, if it exists.

Algebra of Limits :- Let 'f' and 'g' be two functions and 'a' be a point of their common Domain.

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then

1) $\lim_{x\to a} [f(x)+g(x)] = l+m$

2) $\lim_{x\to a} [f(x)-g(x)] = l-m$

3) $\lim_{x\to a} [f(x)\cdot g(x)] = l\cdot m$

4) $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}$ provided $m \neq 0$

Note:- If $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a} g(x)$ doesn't exist.

then $\lim_{x\to a} [f(x)+g(x)]$ doesn't exist.

Note:- If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} f(x)\cdot g(x)$ both exist

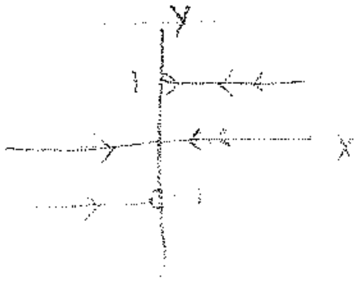
doesnot Necessarily imply that $\lim_{x\to a} g(x)$ also exist.

E.g:- $f(x) = x$ and $g(x) = \frac{|x|}{x}$ so $f(x)\cdot g(x) = |x|$

So Here $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} f(x)\cdot g(x)$ both exist.

and both equal to zero. but $\lim_{x\to 0} g(x)$ doesn't exist.

$$g(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Here L.H.L = −1 and R.H.L = 1 when x approches to 0. Hence limit doesn't exist at $x=0$.

Note:- $\lim_{x\to a} f(x) = l$ then $\lim_{x\to a} |f(x)| = |l|$? converse Not always true.

E.g:- $f(x) = \begin{cases} -1 & ; x < 0 \\ 1 & ; x \geq 0 \end{cases}$

R.H.L = 1 & L.H.L = −1 at $x=0$
So Limit doesn't exist of $f(x)$ at $x=0$, But $|f(x)| = 1 \ \forall x \in \mathbb{R}$

and $\lim_{x\to 0} |f(x)|$ exist and equal to 1.

Squeeze Principal:- Let $f(x) \leq g(x) \leq h(x)$ and

$\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x)$ exist and equal to l

Result:- if $\lim_{x\to a} f(x) = l$, then f is bdd on some deleted Nbd of $x=a$

Result:- if $\lim_{x\to a} f(x) = l$ and $l > 0$ then $\exists$ a δ

$f(x) > 0$ for $0 < |x-a| < \delta$.

Result:- if $\lim_{x\to a} f(x) = l$ and $l > 0$ then $\exists$ a $\delta > 0$ s.t $f(x) > 0$ for $0 < |x-a| < \delta$

Result: If $f(x) \geq 0 \ \forall \ x \in$ Domain and $\lim_{x \to a} f(x) = l$ then $l \geq 0$

Result If $f(x) \leq g(x)$ and $\lim_{x \to a} f(x) = l_1$ and $\lim_{x \to a} g(x) = l_2$ then $l_1 \leq l_2$

Result if f is monotonically increasing on (a,b) then for each $c \in (a,b)$ we get

$$\boxed{\lim_{x \to c^-} f(x) = \sup_{x \in (a,c)} f(x) \leq f(c) \leq \inf_{x \in (c,b)} f(x) = \lim_{x \to c^+} f(x)}$$

Ques:- find $\lim_{x \to -\infty} [x^2 \operatorname{sig}(\cos x)]$

Let $x_n = -2n\pi$ so $\langle x_n \rangle \longrightarrow -\infty$ as $n \to \infty$

$f(x_n) = (-2n\pi)^2 \cdot \operatorname{sig}[\cos(2n\pi)] = 4n^2\pi^2 \operatorname{sig} \cos 2n\pi$
$= 4n^2\pi^2$

$\lim_{n \to \infty} f(x_n) = \infty$

Again let $x_n' = -(2n+1)\pi$ $\{\cos(2n+1)\pi = -1 \ \forall \ n \in \mathbb{Z}\}$
$\langle x_n' \rangle \longrightarrow -\infty$ as $n \to \infty$

$f(x_n') = (-(2n+1))^2 \cdot \operatorname{sig} \cos(-(2n+1)\pi)$
$= (2n+1)^2 \operatorname{sing}(-1)$
$= -(2n+1)^2$

$\langle f(x_n') \rangle \longrightarrow -\infty$ as $n \to \infty$.

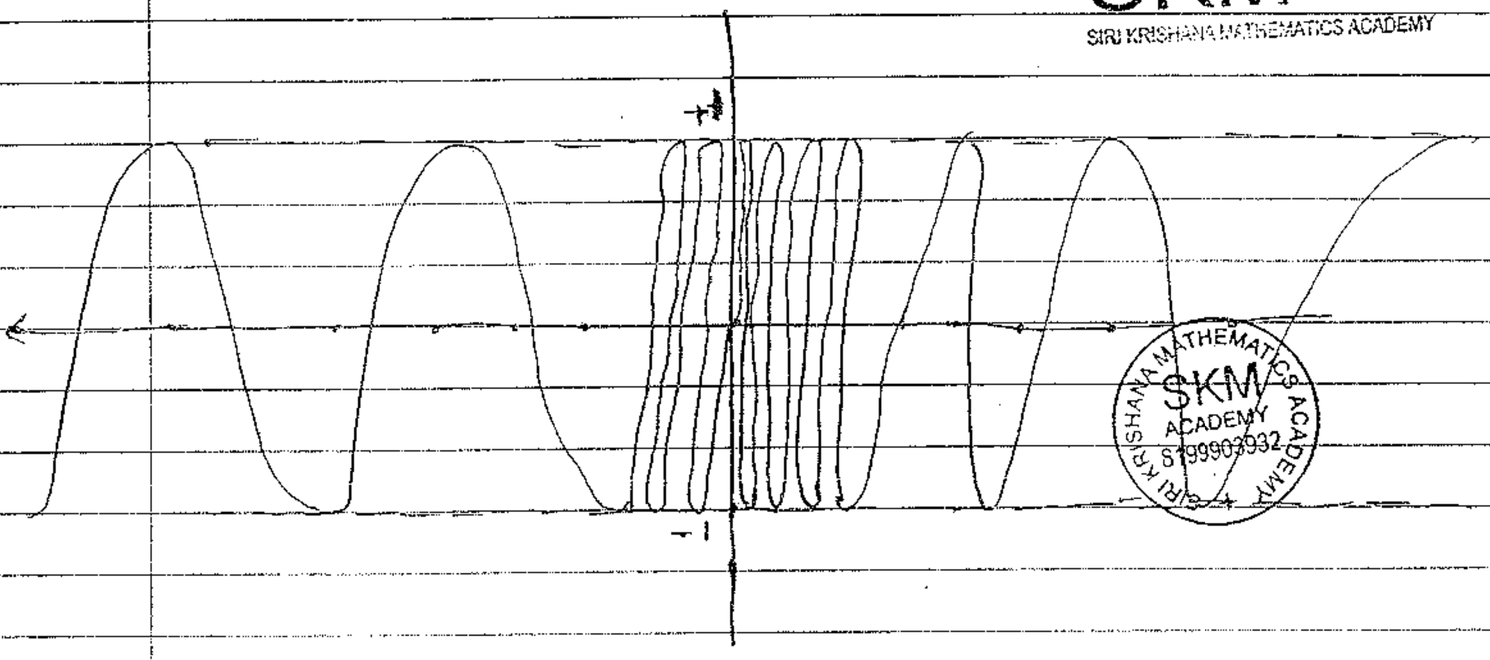
$\Rightarrow \lim_{x \to -\infty} x^2 [\operatorname{sig}(\cos x)]$ does not exist [by Hein's th^m]

Ques:- $\lim_{x\to 0} \sin\frac{1}{x}$ doesn't exist in $\mathbb{R}$.

Let $f(x) = \sin\frac{1}{x}$ then if $x = \frac{1}{n\pi}$ $(n \in \mathbb{Z})$ $f(x) = 0$

~~Because $f\left(\frac{1}{n\pi}\right) = \sin n\pi = 0 \quad \forall\ n \in \mathbb{Z}$~~

and graph of $f(x) = \sin\frac{1}{x}$ is



By inspection graph, when we approch near to $x=0$ graph becomes parallel to y between $+1$ and -1 and oscillate many terms (infinite) between $+1$ and -1. So when $x \to 0$, $f(x)$ never reach at a fixed point, it oscillates so limit doesn't exist. Now by the defination of R.H.L & L.H.L

$$\underset{x\to 0^+}{f(x)} = \lim_{h\to 0^+} f(0+h) = \lim_{h\to 0^+} \sin\frac{1}{0+h} = \sin\infty$$

= any finite Number b/w +1 & −1

= oscillation b/w 1 & −1

So $\lim_{x \to 0^+} f(x)$ doesn't exist.

Similarily $\lim_{x \to 0^-} f(x)$ doesn't exist, Hence In all

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin \frac{1}{x}$ doesn't exist in $\mathbb{R}$.

Que:- Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

We will prove the limit of function at $x=0$ by sequeeze principle.

$$x - \frac{1}{6}x^3 \leq \sin x \leq x \quad \text{for } x \geq 0 \quad \text{— (i)}$$

and $$x \leq \sin x \leq x - \frac{1}{6}x^3 \quad \text{for } x \leq 0 \quad \text{— (ii)}$$

From (i) and (ii) we have conclusion

$$1 - \frac{1}{6}x^2 \leq \frac{\sin x}{x} \leq 1 \quad \text{for } \forall\, x \in \mathbb{R} - \{0\}$$

$\lim_{x \to 0} \left(1 - \frac{1}{6}x^2\right) = 1$ and $\lim_{x \to 0} (1) = 1$ so by squeeze principle

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Que:- Find L.H.S and R.H.S of $f(x) = e^{1/x}$ at $x=0$ and with the help of L.H.L and R.H.L draw a curve in $y = e^{1/x}$

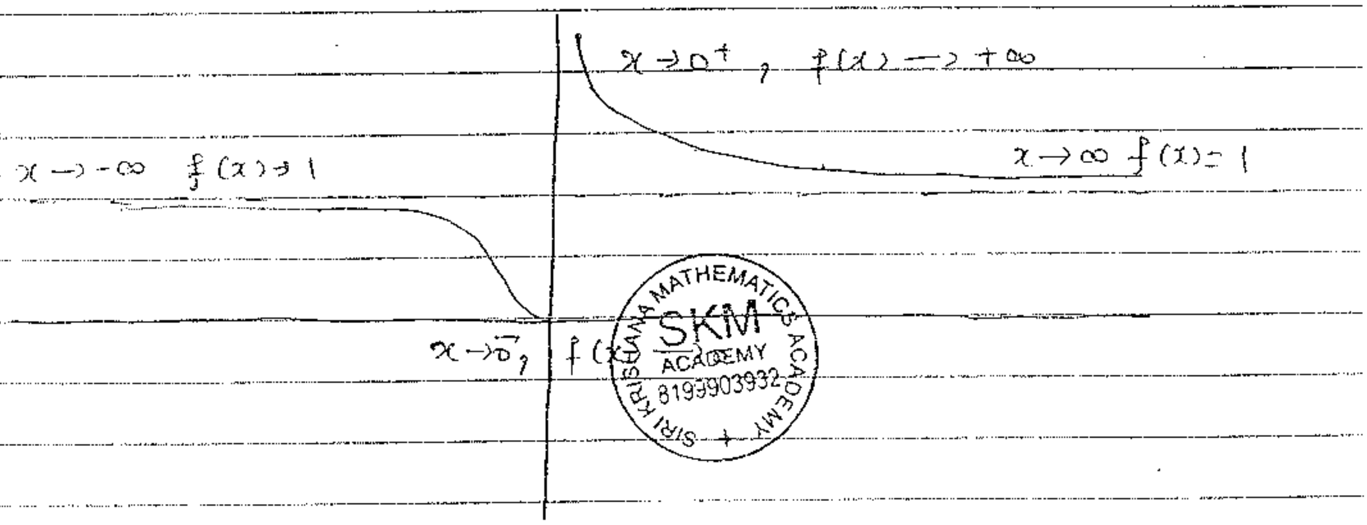
R.H.L $\lim_{x\to 0^+} f(x) = \lim_{h\to 0^+} f(0+h) = \lim_{h\to 0^+} e^{\frac{1}{h}} = \infty$

L.H.L $\lim_{x\to 0^-} f(x) = \lim_{h\to 0^-} f(0-h) = \lim_{h\to 0^-} e^{-1/h} = 0$

Here L.H.L and R.H.L are Not equal So limit does Not exist.

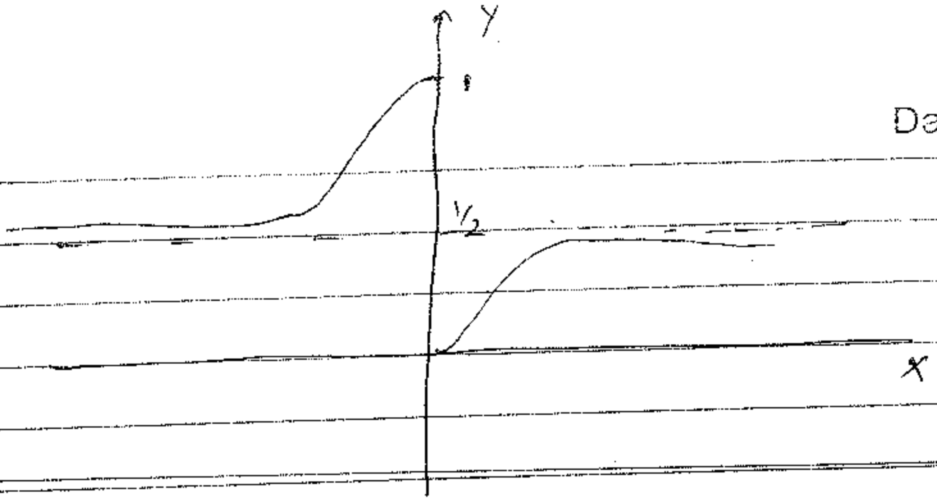
Now at $x=0$; $f(x)=1$ and when $x\to\infty$
$f(x)=1$ and $x\to-\infty$ $f(x)=1$

Now from L.H.L and R.H.L at $x=0$ and by Some other observation or behaviour of $f(x)$ when $x\to\infty$ and $x\to-\infty$, we can draw a curve for $e^{1/x}$



Above graph is of $e^{1/x}$ for $x\neq 0$

Que: Draw graph and L.H.L and R.H.L at different points of function $y = \frac{1}{e^{1/x}+1}$



graph of $\frac{1}{e^{1/x}+1}$

(Do your self)

Continuity

Continuity :-

Let $A \subseteq \mathbb{R}$, c is point of A, Function is said to be continuous at c for every $\epsilon > 0$ $\exists$ δ such that

$$|f(x) - f(c)| < \epsilon \text{ when } |x - c| < \delta.$$

or

$$\lim_{x \to c} f(x) = f(c)$$

or

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

Def^n:- Let $S \subseteq \mathbb{R}$, and 'f' be a function defined on S (ie $f : S \longrightarrow \mathbb{R}$), let any '$a$' $\in S$, then 'f' is said to be continuous at $x = a$ in any of the following condition

i) 'a' is an isolated member of S.

ii) if 'a' is limit point of S then $\lim_{x \to a} f(x) = f(a)$

Notes:- Every function is ~~isot~~ continuous at isolated points of its Domain.

Notes:- If 'a' is Not Isolated point i.e 'a' is limit point then three condition must hold for f to be continuous at $x = a$,

i) f must be defined at a (ie $f(a)$ is meaningful)

ii) the limit of f at $x = a$ must exist in $\mathbb{R}$

iii) These two values must be equal ($\lim_{x \to a} f(x) = f(a)$)

Second Defination :- Let $f : S \to \mathbb{R}$ be a function, Let $a \in S$, f is cts at $x = a$ if given any Number $\epsilon > 0$, $\exists$ $\delta > 0$ s.t if $x \in (a-\delta, a+\delta)$, then $|f(x) - f(a)| < \epsilon$

if 'f' fails to be continuous at $x = c$, we say f is discontinuous at $x = c$.

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Note :- A function 'f' is cts from Right at $x = a$ iff $\lim_{x \to a^+} f(x) = f(a)$ [i'e R.H.L at 'a' = f(a)]

Note :- A function 'f' is cts from left at $x = a$ iff $\lim_{x \to a^-} f(x) = f(a)$ [i'e L.H.L at 'a' = f(a)]

Note :- clearly, f is cts at $x = a$ iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

Note :- 'f' is continuous on $S (\subseteq \mathbb{R})$ if function is continuous at every point of S.

Note :- function 'f' is said to be continuous in $[a, b]$ if it is

i) Right continuous at 'a' i'e $\lim_{x \to a^+} f(x) = f(a)$

ii) continuous in (a, b)

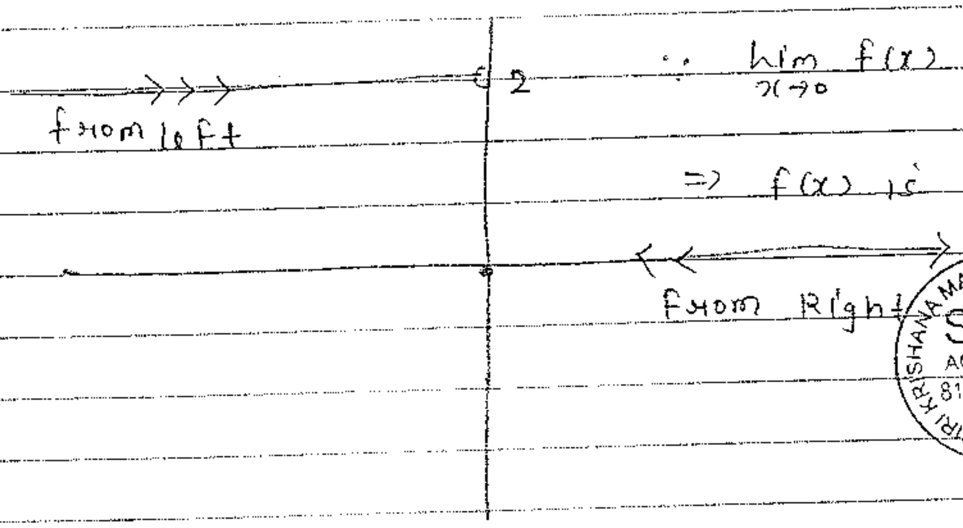
iii) Left continuous at 'b' i'e $\lim_{x \to b^-} f(x) = f(b)$

Note :- A continuous function Means a function whose graph is without a break or ~~intersect~~ interruption

Eg:- $f(x) = \begin{cases} \frac{x - |x|}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

i.e $f(x) = \begin{cases} \frac{x - x}{x} = 0 & \text{if } x > 0 \\ \frac{x - (-x)}{x} = 2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

So graph of f(x) is



$\therefore \lim_{x \to 0} f(x)$ does Not exist

$\Rightarrow f(x)$ is Not cts at $x=0$

By graph we can say f is Not cts at $x=0$, when we approch to '0' from Right and left, f(x) approches 0 and 2 Respectively. Hence $\lim_{x \to 0} f(x)$ doesn't exist.

Eg:- $f : [0,1] \longrightarrow \mathbb{R}$

$f(x) = \begin{cases} x & x = 0 \\ -x & x \in (0,1) \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$

function is cts at $x = 0$.
our function is cts at $[0,1)$.

Eg 1 :- $f : (0,1) \longrightarrow \mathbb{R}$
$f(x) = \frac{1}{x}$.

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is function is cts at $x = 1$?
at $x = 1$ function is Not defined since $1 \notin (0,1)$
$\Rightarrow$ function is discts.

Note :- Let $c \in A$ is limit point of A, function f is said to be cts at c if $\lim_{x \to c} f(x) = f(c)$.

2) Let c is limit point of A, function f is said to be cts at c, if function must defined at c and lim of f at c exist and equal
i.e $\lim_{x = c} f(x) =$ exist and equal.

3) if c is isolated point then function always cts.

4) Every Seqn is continuous

5) $f : \mathbb{Z} \longrightarrow \mathbb{R}$ always cts.

Eg: $f : [0,1) \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 2/x & ; \ x \neq 0 \\ x & ; \ \text{otherwise} \ x = 0 \end{cases}$$

$f(0) = 0$

$\lim_{x \to 0^-} f(x) = \infty$

$\lim_{x \to 0^+} f(x) = \infty$.

function is discontinuous at $x = 0$

$\Rightarrow$ fun. is discontinuous on $[0,1)$.

Note: Every seqn of Real Number is cts function.

Examples of continuous function:-

1) $f(x) = \sin x$

2) $f(x) = \cos x$

3) $f(x) = P(x)$ $\left[P(x) = \sum_{i=0}^{n} a_i x^i \quad a_i \right]$

4) $f(x) = e^x$

5) $\sin x + \cos x$, $\sin x + P(x)$
$\sin x + e^x$, $\cos x + P(x)$
$\cos x + e^x$, $P(x) + e^x$

6) $\sin x - \cos x$, $\sin x - P(x)$
$\sin x - e^x$, $\cos x - P(x)$
$\cos x - e^x$, $P(x) - e^x$

7) $\sin x \cdot \cos x$, $\sin x \cdot P(x)$
$\sin x \cdot e^x$, $\cos x \cdot P(x)$
$\cos x \, e^x$, $P(x) \cdot e^x$

8) $\frac{\sin x}{e^x}$, $\frac{\cos x}{e^x}$, $\frac{P(x)}{e^x}$

1) Domain of Defination of all examples is $\mathbb{R}$

2) All are continuous function on $\mathbb{R}$

3) Here $\mathbb{R}' = \mathbb{R}$ so for any $a \in \mathbb{R}$ $\lim_{x \to a} f(x) = f(a)$

4) $P(x)$ = Poly. function.

Some other functions:-

9) $f(x) = [x]$, greatest integer

dod = $\mathbb{R}$

Domain of continuity $(D_c) = \mathbb{R} - \mathbb{Z}$

i.e $f(x)$ is Not cts at any point of Integer.

i.e for any $\alpha \in \mathbb{Z}$

$\lim_{x \to \alpha^+} f(x)$ exist and also $\lim_{x \to \alpha^-}$ exist but

R.H.L $\neq$ L.H.L.

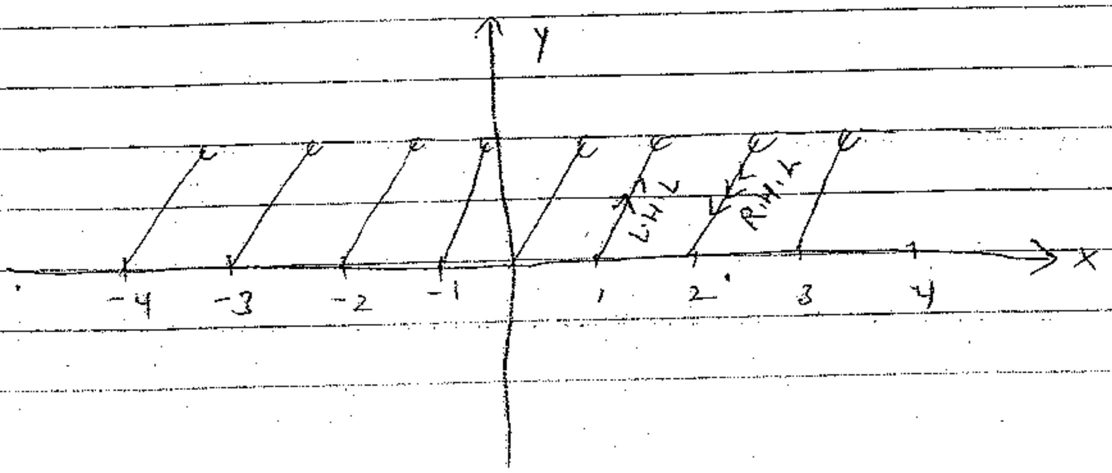
So finaly $\lim_{x \to \alpha} f(x)$ doesn't exist so function

is cts at every point of $\mathbb{R}$ except integers.

10) $f(x) = \{x\}$, fractional part function.

Domain of defination (dod) = $\mathbb{R}$

Domain of continuity $(D_c) = \mathbb{R} - \mathbb{Z}$

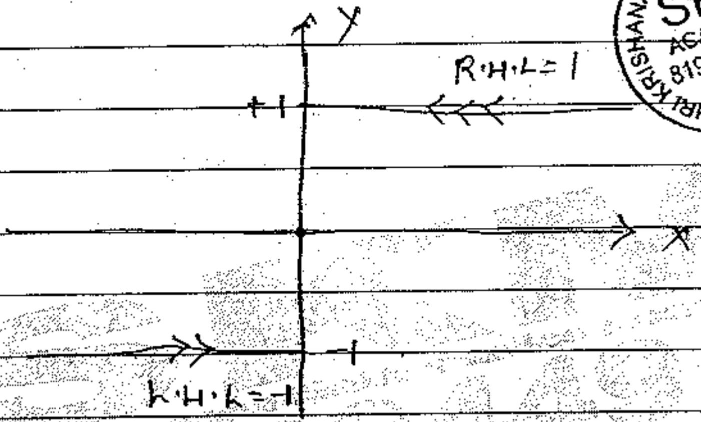


Except integer both R.H.L and L.H.L exist and equal to value of function but at integer L.H.L and R.H.L both exist but Not equal so limit doesn't exist at integer.

E.g. at $x=2$ L.H.L (>>) at $x=2 = 2$
R.H.L (>>>) at $x=2 = 0$

both exist but Not equal.

# $$f(x) = \text{sig}(x) = \begin{cases} \frac{|x|}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$



Dod = $\mathbb{R}$
$D_c = \mathbb{R} - \{0\}$.

By looking the graph, R.H.L & L.H.L at '0' both exist but Not equal so function is Not cts only at $x=0$.

$\text{Sig}(f(x)) = \begin{cases} \dfrac{|f(x)|}{f(x)} & ; \ f(x) \neq 0 \\ 0 & ; \ f(x) = 0 \end{cases}$

Domain of Defination will be same as Domain of defination of f(x). Let $D_{od} = D_f$

Domain of Continuity (D_c) of $\text{Sig}(f(x)) =$

$$D_f - \{x \in D_f ; f(x) = 0\}$$

i'e $\text{Sig}\{f(x)\}$ will be continuous at every point of Domain of f(x) except those points where f(x) is zero.

$\text{Sing}(\sin x) = \begin{cases} -1 & ; \ \sin x < 0 \\ 1 & ; \ \sin x > 0 \\ 0 & ; \ \sin x = 0 \end{cases}$

$D_{od} = \mathbb{R}$

i'e $D_f = \mathbb{R}$

Domain of continuity $(D_c) = D_f - \{x \in D_f , f(x) = 0\}$

$= \mathbb{R} - \{x \in \mathbb{R} ; \sin x = 0\}$

$= \mathbb{R} - \{n\pi ; n \in \mathbb{Z}\}$

i'e Sig (sinx) will be cts on $\mathbb{R}$ except $n\pi$ type points where $n \in \mathbb{Z}$.

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$f(x) = \frac{1}{2}x^2 + \cos x + \log x$

dod = (dod of $\frac{1}{2}x^2$) ∩ (dod of $\cos x$) ∩ (dod of $\log$

$= \mathbb{R} \cap \mathbb{R} \cap (0,\infty)$

$= (0,\infty)$

$D_c = (0,\infty)$

$f : \mathbb{R} \to \mathbb{R}$ s.t

$$f(x) = \begin{cases} a & ; x \in \mathbb{Q} \\ b & ; x \in \mathbb{Q}^c \end{cases}$$

Then this typ. function f(x) is called Dirichlet's function [General Dirichlet's function]. Here a and b are any two const from $\mathbb{R}$ (Set of Real Number).

If you choose any 'a' and 'b' from $\mathbb{R}$, that will be your Dirichlet's function.

Note:- Dirichlet function Nowhere cts i.e $D_c = \phi$.

2014. $f(x) = \begin{cases} \sin 1/x & ; x > 0 \\ x & ; x < 0 \end{cases}$

Domain of defination = $\mathbb{R}$

Let $a_n = \frac{1}{n\pi} < \delta \to 0$, $b_n = \frac{1}{(4n-3)\frac{\pi}{2}} < \delta \to 0$

where $a_n \in (0, 0+\delta)$ and $b_n \in (0, 0+\delta)$

$f(a_n) = 0$ & $f(b_n) = 1$

$\lim_{n\to\infty} f(a_n) = 0$ and $\lim_{n\to\infty} f(b_n) = 1$

So R.H.L doesn't exist (By Hein's Thm).

But if we choose any $\langle a_n \rangle$ from $(0-\delta, 0)$ which tends to '0', $f(a_n)$ always tends to '0'. So L.H.L exist and equal to zero

In all limit doesn't exist at $x=0$.
So function $f(x)$ is cts at every point of dod except $x=0$ so $d_c = \mathbb{R} - \{0\}$.

Q. $f(x) = \begin{cases} p \sin\frac{1}{q} & ; x = \frac{p}{q} \text{ in } [0,1] \\ x & ; x = 0 \text{ or } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$

dod $= [0,1]$.

Now for limit :-
we can construct always such seqn which are converges at some point b

i.e $a_n, b_n \in \left(b - \frac{1}{n}, b + \frac{1}{n}\right)$ s.t

$a_n = \frac{p_n}{q_n} \in \mathbb{Q}$, $b_n \in \mathbb{Q}^c$

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and $\langle a_n \rangle \longrightarrow b$ and $\langle b_n \rangle \longrightarrow b$

So $f(a_n) = p_n \cdot \sin \frac{1}{q_n}$

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \left(p_n \cdot \sin \frac{1}{q_n} \right) = \lim_{n\to\infty} \frac{p_n}{q_n} \left(\frac{\sin 1/q_n}{1/q_n} \right)$$

$$= \lim_{n\to\infty} \frac{p_n}{q_n} = b$$

and $\lim_{n\to\infty} f(b_n) = b$

So function's limit exist everywhere. But Not cts at Rationals. b'coz from above, we see that $\lim_{n\to\infty} f(a_n) \neq F(a)$ where 'a' is Rational. and

if we take any $\langle b_n \rangle$ which is converges to an irrational number say 'c'. $f(b_n)$ always converges to 'c'. and the value of function at 'c' is $f(c) = c$.

So function will be continuous at $x=0$ and at irrationals.

So Domain of continuity $(D_c) = \{[0,1] \cap \mathbb{Q}^c\} \cup \{0\}$.

(Q3) $f : [0,1] \longrightarrow \mathbb{R}$ st

$$f(x) = \begin{cases} 1/q & ; \ x = p/q \text{ in } [0,1] \\ 0 & ; \ x = 0 \text{ or } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Limit exist at every point and limit is zero and fun. is cts only at Irrationals at 30P But not cts at Rationals [Same as Previous example].

$D_c = \{[0,1] \cap Q^c\} \cup \{0\}$

Note $f(x) = \begin{cases} p(x) , & x \in Q \\ q(x) , & x \in Q^c \end{cases}$

then f(x) is cts at $x = \alpha$ if $p(\alpha) = q(\alpha)$ i.e if α is Root of $p(x) - q(x) = 0$, then f(x) is cts at $x = \alpha$ and these are the only points at which function will be cts.

Eg:- $f(x) = \begin{cases} \sin x ; & x \in Q \cap [0,1] \\ \cos x ; & x \in Q^c \cap [0,1] \end{cases}$

So f(x) is cts only at $x = \frac{\pi}{4}$

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Qu Find the conditions on p and q s.t

$f(x) = \begin{cases} x^p \sin \frac{1}{x^q} ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$

(i) Limit does Not exist at 0
(ii) f(x) is cts in $\mathbb{R}$

Solution:→ $f(x)$ is cts everywhere except at $x=0$, now at $x=0$ $\lim_{x\to 0} x^p \sin\frac{1}{x^q} = \lim_{x\to 0} \frac{\sin 1/x^q}{1/x^p}$

$$= \lim_{x\to\infty} \frac{\sin x^q}{x^p}$$

$$= \lim_{x\to\infty} x^{q-p} \frac{\sin x^q}{x^q}$$

$$\lim_{x\to\infty} x^{q-p} \cdot 0$$

So limit doesn't exist when $q-p>0 \Rightarrow \boxed{p<q}$ and limit will exist and equal to 0 when $q-p<0 \Rightarrow \boxed{p>q}$

So when $p>q$ fun. will be cts at $x=0$.

Types of Discontinuity:-

1.) Removable Discontinuity :- $A \subseteq \mathbb{R}$; $f : A \to \mathbb{R}$ and function is discontinuous at c is said to be Removable Discontinuity at c if L.H.L = R.H.L $\neq f(c)$

e.g.:- $f(x) = \begin{cases} x & ; x \in \mathbb{R} - \{1\} \\ 1000 & ; x = 1 \end{cases}$

L.H.L = R.H.L $\neq f(c)$

2> **First Kind or Jump Discontinuity**

$A \subseteq \mathbb{R}$; $f : A \to \mathbb{R}$ and function is discontinuous at c is said to be first kind discontinuity at c if L.H.L and R.H.L exist but Not equal i.e R.H.L ≠ L.H.L

Signum function is Example of first kind Discontinuity.

Eg:-
$$f(x) = \begin{cases} 1 & ; \quad x > 0 \\ -1 & ; \quad x < 0 \\ 0 & ; \quad x = 0 \end{cases}$$

L.H.L ≠ R.H.L

Eg:- $f : (0,2) \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ x+1 & 1 < x < 2 \end{cases}$$

at $x = 1$ function Not Defined.
function is Not cts on $(0,2)$

Eg:- if
$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ x+1 & 1 < x < 2 \\ 1 & x = 1 \end{cases}$$

then function is discontinuity of first kind

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2nd kind discontinuity :- $A \subseteq \mathbb{R}$, $f : A \longrightarrow \mathbb{R}$; function is discontinuity at c if Neither L.H.L Nor R.H.L exist

or $\lim_{x \to c^-} f(x)$ does Not exist

or $\lim_{x \to c^+} f(x)$ does Not exist

or $\lim_{x \to c^-} f(x)$ does Not exist and $\lim_{x \to c^+} f(x)$ does Not exist.

Result :- Let $I = [a,b]$ be closed and bdd interval and function is cts on I then function f is bdd on I

Converse of this Result is Not true

Que $[a, \infty)$ is

1) closed interval
2) Closed set
3) closed and unbdd set
4) closed and bdd interval
5) unbdd set

# $[a,b]$ infinite set bdd.

$(a,b]$ "

$[a,b)$ "

$[a,\infty)$ infinite set unbdd.

(a,∞) "

$(-\infty,a)$ "

$(-\infty,a]$ "

(a,b) infinite bdd

$[a,a] = \{a\}$ Degenerate

$(a,a] = \phi$

$[a,a) = \phi$

$(a,a) = \phi$

Result:- A Monotone function is bdd if defined on closed Interval

E.g $\frac{1}{x}$; $x \in [0,1]$, Not defined at $x=0$

then $\frac{1}{x}$; $x \in (0,1)$.

i.e $f(x)=\frac{1}{x}$ is Not bdd on $[0,1]$.

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Eg:- $f : [0,\infty) \longrightarrow \mathbb{R}$

$f(x)=x$

function is unbdd.

function is cts on closed set $[0,\infty)$.

But function is unbdd on $I=[0,\infty)$.

Eg:- $f : [0,\infty) \longrightarrow \mathbb{R}$

$$f(x)=\frac{1}{x^2+1}$$

function is bdd on $I=[0,\infty)$. But I is unbdd. (function is Monotone and defined on $[0,\infty]$ $\Rightarrow$ function is bdd).

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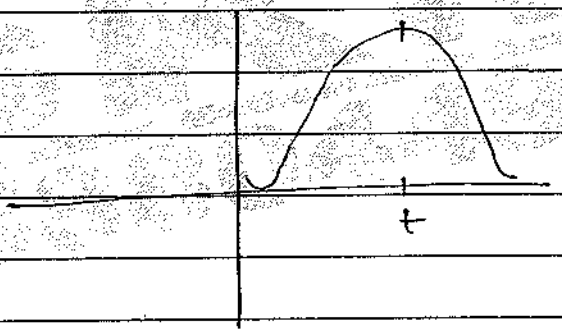
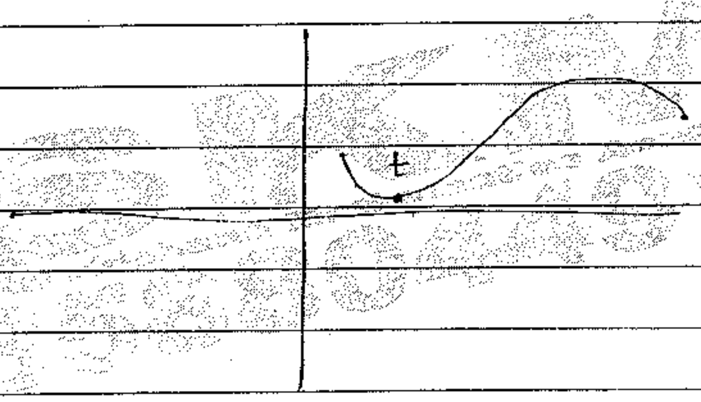
# Unbdd:- A function f is unbdd for each +ve integer m, $\exists$ a $x \in A$ s.t $|f(x)| \geq M$.

Boundness thm :- Let $I = [a,b]$ be called closed and bounded Interval and function is cts then function has absolute maximum and Absolute Minimum.

Absolute Maximum :- Let I be an Interval closed and bdd, function is said to be Absolute Maximum on I if $\exists$ a point $t \in I$ s.t $f(x) \leq f(t)$ $\forall$ $x \in I$



Absolute Minimum :- Let I be an Interval closed and bdd., function is said to be Absolute Minimum on I if $\exists$ a point $t \in I$ s.t $f(x) \geq f(t)$ $\forall$ $x \in I$.



Absolute Maximum $\Rightarrow$ Supermum
But Not conversely

Ex :- $f(x) = x$; $x \in (0,1)$

Sup $= 1$

But Not Absolute Maximum Since it is defined on

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closed and bdd interval.

# Absolute Minimum ⇒ infrimum But Not conversely same Example inf $= 0$ But Not Absolute Minimum.

# Let $I = (a,b)$ be bdd and open interval, f is Monotonically increasing and $f : I \longrightarrow \mathbb{R}$ then

i) if f is Bdd above then $\lim_{x\to b^-} f(x) = f(x) = \sup_{x\in(a,b)} f(x)$

ii) if f is Bdd below then $\lim_{x\to a^+} f(x) = \inf_{x\in(a,b)} f(x)$

iii) if f is unbdd above then $\lim_{x\to b^-} f(x) = \infty$

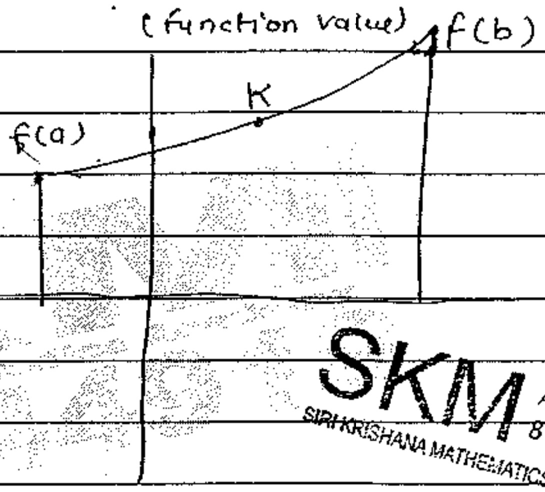
iv) if f is unbdd below then $\lim_{x\to a^+} f(x) = -\infty$

Note:- if function is Monotonic decreasing in above. Result is Reverse.

Bolzano Weirstress theorem:- Let I be closed and bdd interval and $f : I \longrightarrow \mathbb{R}$ cts function and $I = [a,b]$ $f(a)\cdot f(b) < 0$ (or $f(a)$ and $f(b)$ one of the opposite sign) then $\exists$ a $c \in I$ s.t $f(c) = 0$.

Intermediate value thm:- Let $I = [a,b]$, be closed and bdd and $f : I \longrightarrow \mathbb{R}$ is cts, $f(a) \neq f(b)$ then f assume every value b/w $f(a)$ and $f(b)$

#Let $I = [a,b]$ be closed and bdd interval and $f : I \longrightarrow \mathbb{R}$ is cts function. then $f(I)$ is closed and bdd.



#Constant we can ~~writte~~ write in degenerate interval

#Let $I = [a,b]$ closed and bdd interval. Let $f : I \longrightarrow \mathbb{R}$ cts. for every K s.t $f(a) \le K \le f(b)$ then $\exists$ a $c \in (a,b)$ s.t $f(c) = K$.

#Image of compact set under cts function is compact set.

#Image of closed and bdd interval under cts Functio is closed and bdd.

#Same in connected set.

#$f : \mathbb{R} \longrightarrow \mathbb{R}$ cts and $f(1/2) = 1/2$ then $f(1/3) = 1/2$

$f(x)$ is cts on $x \in \mathbb{R}$

Identity theorem :-

#$f : \mathbb{R} \longrightarrow \mathbb{R}$ cts and $f(x)$ is cts $x \in \mathbb{Q}$ then $f(x)$ is cts on $x \in \mathbb{R}$

$I = [a,b]$ closed and bdd interval. Let $f : I \to \mathbb{R}$ cts. then $f(I)$ is either constant or Interval. (i.e interval or Non-Interval).

E.g:- $f : \mathbb{R} \longrightarrow \mathbb{R}$

$f(x) = x$

$f(\mathbb{R}) = \mathbb{R}$ (interval)

Path is cts function $[a,b] \longrightarrow \mathbb{R}$.

I.V.T Property:- $I = [a,b]$ closed and bdd interval and $f : I \longrightarrow \mathbb{R}$ is cts function for every pair x_1, x_2; ~~f(a)~~ $a \le x_1 < x_2 \le b$ for each k, k lies b/w $f(x_1)$ and $f(x_2)$ then $\exists$ a $c \in (x_1, x_2)$ s.t $f(c) = k$.

if function is cts on $[a,b]$ then function has I.V.T Property. But Not conversely.

i.e function has I.V.T Property then Not Need function is cts.

Example of Not converse

$c \in (x_1, x_2)$

Ex:- $f : [-1,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

Function has all value $[-1,1]$. But Not cts at $x=0$.

Let k b/w $f(-1)$ and $f(1)$ then $\exists$ $c \in (-1,1)$ s.t $f(c) = k$

function has IVT Property. But function is Not cts $[-1,1]$.

Note:- Every Strictly Monotone function is always one-one

Monotonicly Increase :- Let $I \subseteq \mathbb{R}$, $f: I \longrightarrow \mathbb{R}$ function is said to be Monotonicly Increase. if $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$

Monotonicaly decreasing:- Let $I \subset \mathbb{R}$; $f : I \longrightarrow \mathbb{R}$ function is said to be Monotonicaly decreasing if $x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$.

Strictly Monotone Increasing:- Let $I \subseteq \mathbb{R}$, $f: I \rightarrow \mathbb{R}$ function is said to be Strictly Monotonicaly increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
or $x_1 \le x_2 \Rightarrow f(x_1) < f(x_2)$

Note:- if function is S.M.I on A. $A \subseteq \mathbb{R}$. $f: A \longrightarrow \mathbb{R} \Rightarrow f$ is one-one. But Not conversely.

i.e $f : A \xrightarrow{S.M.I} \mathbb{R} \Rightarrow f$ is one-one

Let $x_1, x_2 \in A$

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

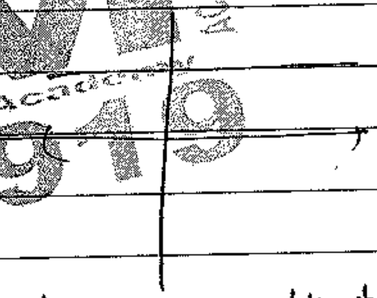
$\Rightarrow$ function is one-one.

Note:- if function is M↑ on A. $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ then Not Needed function is one-one.

Ex:- Constant function which is M↑ but Not one-one.

$f: A \longrightarrow \mathbb{R}$

$f(x) = K$.

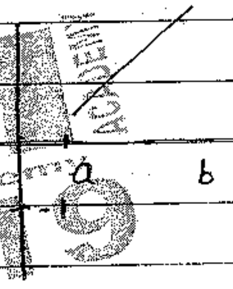


cut more than one point ⇒ Not one-one.

function is one-one But Not S.M increasing.

Ex:- $f: [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x & ; x \in [0,1) \\ -1 & ; x = 1 \end{cases}$$



Note:- Let $I = [a,b]$ be closed and bdd interval. $f: I \longrightarrow \mathbb{R}$ has I.V.T Property if f is one-one then f is S.M.I

Note: Let $I = [a,b]$ be closed and bdd interval. $f: I \longrightarrow \mathbb{R}$ and function has I.V.T Property, f is injective then f is cts.

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Note:- Let $I = [a,b]$ be closed and bdd interval and function is cts, if f is one-one $\Rightarrow$ f is S.M.T.

Note Let $I = [a,b]$ be closed and bdd Interval and $f : I \to \mathbb{R}$, and function has I.V.T Property if function f is S.M.T then function is cts.

i.e S.M.I on A $\longrightarrow$ 1-1

S.M.I and cts.

S.M.I $\Longleftrightarrow$ 1-1

ii) cts on $[a,b]$ $\longrightarrow$ I.V.T.P

S.M.I and I.V.T $\Rightarrow$ cts.

S.M.I and Injective $\Rightarrow$ cts.

Note:- if f is cts on $\mathbb{R}$ then we can't say that It will attain sup. and Inf or Not.

E.g:- $f(x) = \left\{ \frac{1}{1+|x|} \right\}$ $\forall$ $x \in \mathbb{R}$

It is cts on $\mathbb{R}$ and attains its sup at $x=0$ but doesn't attains Inf.

Note:- if function 'f' is continuous at $x=a$ and $f(a) \neq 0$; then $f(x)$ Retain its sign as that of $f(a)$ in some Nbd of 'a' i.e $f(x)$ has same sign as $f(a)$ in some Nbd of 'a'.

Note:- Under Non-constant continuous function, the image of interval is closed interval.

i.e. 'f' be a function defined on $[a,b]$ then the Range of continuous function 'f' is closed interval and if 'f' is constant then singlton.

Note:- if Domain is Not closed interval then Range set may or may Not be closed interval.

Note:- $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$

if $f(\mathbb{R}) \in \mathbb{Q}$

$\in \mathbb{Q}^c$

$\in$ Countable set

$\in$ Algebraic set

$\in$ Transcedental

$\in \mathbb{N}$

$\Rightarrow$ f is constant function.

Note:- if function 'f' is continuous on $[a,b]$ and $f(a) = f(b)$ then $\exists$ $x_1, x_2 \in (a,b)$ s.t $f(x_1) = f(x_2)$

Note:- Identity Theorem (Second form):-

Let 'f' and 'g' are two continuous functions on $\mathbb{R}$.

$f, g : \mathbb{R} \xrightarrow{cts} \mathbb{R}$ s.t

$f(x) = g(x)$ $\forall$ $x \in \mathbb{Q}$

then $f(x) = g(x)$ $\forall$ $x \in \mathbb{R}$.

i.e. if $f(x)$ and $g(x)$ are agree on $\mathbb{Q}$ then they will agree on $\mathbb{R}$, if both are continuous on $\mathbb{R}$.

Uniform continuity

Lipscitz condition :- Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $\exists$ constant k, $k > 0$ s.t

$$|f(x_1) - f(x_2)| \le k|x_1 - x_2| \quad \forall \; x_1, x_2 \in A.$$

then function is said to be satisfy Lipscitz condition on A.

(function is satisfied the Lipscitz condition for every point of A.)
where k is Lipscitz constant.

Uniform continuous :- $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ is said to be uniform continuous if for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ $\forall$ $x_1, x_2 \in A$. (δ depends on ϵ only.)

Note:- δ dependent only on ϵ, independent from x.

Note:- In continuity δ depends ϵ and point x $(x \in A)$
But In uniform continuity δ depends only ϵ, Not on point.

Note:- U.C $\Rightarrow$ Continuity, But Not conversely.

Ex:- $f(x) = \begin{cases} \dfrac{1}{x} & \text{on } x \in (0,1) = A \end{cases}$

$f(x)$ is unbdd in interval A.

function is cts on A. But Not U.C on A.
(In Uniform continuous Problem boundary point)

Note: Every Lipscitz function is uniform continuous on A. But Not conversely.

Ex:- $f(x) = x^2$ on $[a,b]$
function is Lipscitz function.

Let $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |(x_1 - x_2)(x_1 + x_2)|$
$= |x_1 - x_2| \, |x_1 + x_2|$

$$\left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right| = |x_1 + x_2| \le K$$

$\Rightarrow$ function is Lipscitz function.
$\Rightarrow$ function is Uniform continuous on A.

Eg:- $f(x) = \sqrt{x}$; $x \in [0,2]$
check function is Lipscitz function or Not.

$$|f(x_1) - f(x_2)| = |\sqrt{x_1} - \sqrt{x_2}| = \left| \frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}} \right|$$

$$|f(x_1) - f(x_2)| \le |x_1 - x_2| \cdot \frac{1}{|\sqrt{x_1} + \sqrt{x_2}|} < K \quad \left[\begin{array}{l} x_1, x_2 \to 0^+ \\ \frac{1}{0} = \infty < K \text{ Not possible.} \end{array} \right.$$

there does Not exist $K \in [0,2]$ satisfied this condition.
$\Rightarrow$ function is Not Lipscitz function.
But this function $f(x) = \sqrt{x}$; $x \in [0,2]$ is uniform continuous.

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# Let $I=[a,b]$ closed and bdd interval. Function is cts on $I \Rightarrow$ function is uniform continuous.

Ex:- $f(x)=\sqrt{x}$; $x \in [0,2]$.
function is cts on $[0,2]$
$\Rightarrow$ function is U.C

i.e $f: [a,b] \xrightarrow{cts} \mathbb{R} \Rightarrow$ U.C.

Result:- Let $I=(a,b)$, $f: I \longrightarrow \mathbb{R}$ is cts, then f is U.C iff $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

Note:- if $I=(a,b)$; $f: I \longrightarrow \mathbb{R}$ is cts then f is Not U.C if one of the $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ does Not exist.

Eg: ~~$f(x)=$~~ $f: (0,1) \longrightarrow \mathbb{R}$
$f(x)=\frac{1}{x}$; $x \neq 0$.

$f(x)$ is cts on $(0,1)$
But $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ Not exist.

$\Rightarrow$ function is Not uniformly continuous

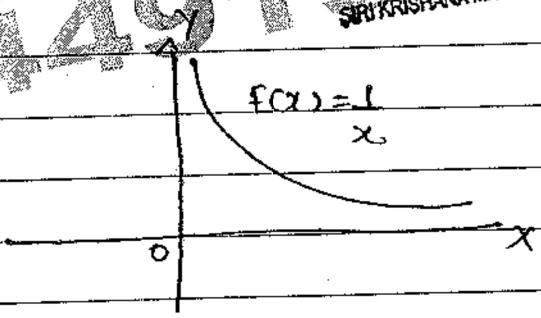
Eg:- $f(x)=\sin\frac{1}{x}$; $x \in (0,1) = A$

function is cts at A
But Not U.C on A
Since $\lim_{x\to 0^+} f(x) =$ Not exist.

Important Note :- Through δ-ε defination, we have some conclusion about the uniform continuity through of functions.

1) if the graph of function become parallel to y-axis in the given interval then f(x) will Not be u.c on that interval.

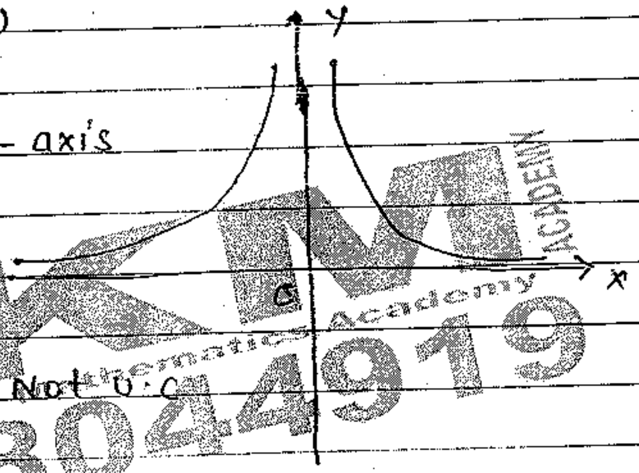
Eg:- $f(x)=\frac{1}{x}$ on $(0,1)$



Graph of f(x) become Parallel as we approch close to 0

Hence $f(x)=\frac{1}{x}$ is Not u.c on $(0,1)$.

Eg:- $f(x)=\frac{1}{x^2}$ on $(0,\infty)$



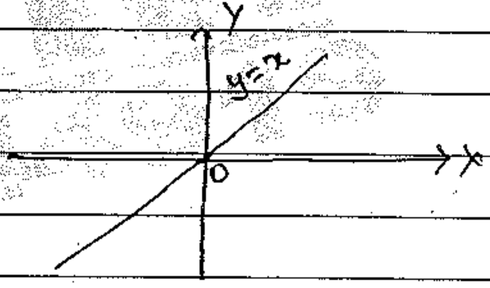
Graph parallel to y-axis as we approach close to 0 from R.H.S or L.H.S

Hence $f(x)=\frac{1}{x^2}$ is Not u.c on $(0,\infty)$ or on $(-\infty,0)$

$f(x)=\frac{1}{x}$ or $f(x)=\frac{1}{x^2}$, graph of these ~~interval~~ function become Parallel to y axis on every ~~that~~ interval which contain zero as end points. Hence we can say, these both functions will be u.c on Every that interval which doesn't zero as end point.

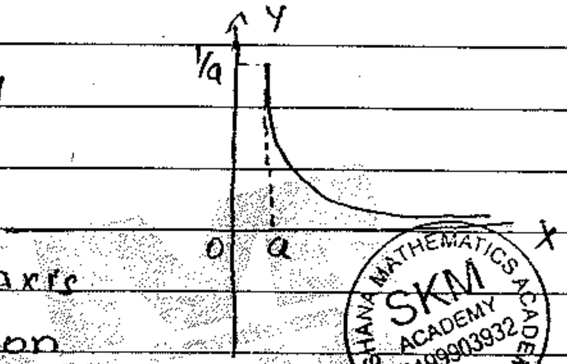
2 if the graph of $f(x)$ become Parallel to x-axis or goes to $+\infty$ or $-\infty$ slowely as graph of $y=x$ line, then $f(x)$ will be u.c.

Eg:- $f(x) = x$,
Graph of x approches to ∞ as $x \to \infty$ very slow. Hence $f(x) = x$ is u.c on $\mathbb{R}$.



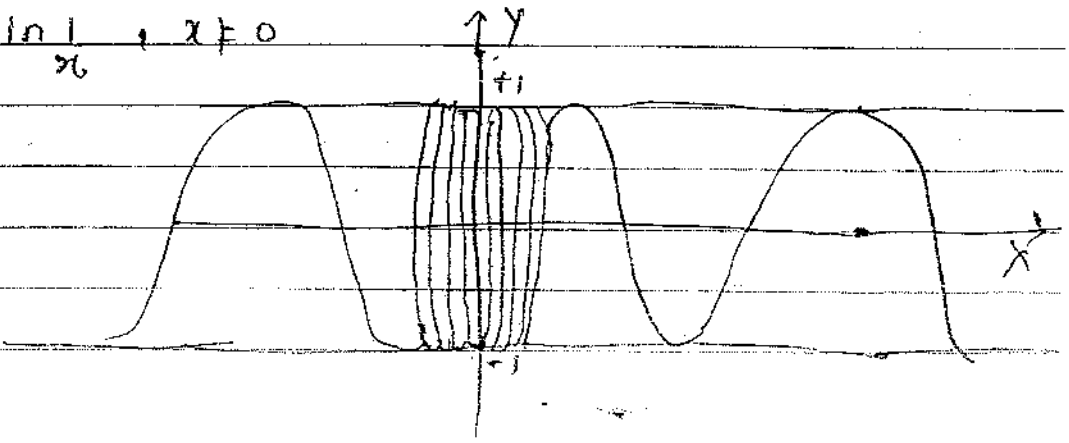
Eg:- $f(x) = \frac{1}{x}$ on (a, ∞), $a > 0$

Graph of $\frac{1}{x}$ is Not parallel to y-axis on (a,∞), $a>0$. And become Parallel to x-axis as $x \to \infty$, Hence u.c on (a,∞) $a>0$.



3) if graph of $f(x)$ oscillate or fluctuate b/w two point many times Parallel to y-axis then $f(x)$ will Not be u.c.

Eg:- $y = \sin\frac{1}{x}$, $x \neq 0$



Graph of $\sin\frac{1}{x}$ Parallel to y-axis and oscillates infinitely b/w 1 and -1, Hence

$f(x) = \sin\frac{1}{x}$ is Not U.C on $(0,1)$.

$f(x) = \sin\frac{1}{x}$ is ~~Not~~ U.C on Every that interval which doesn't contain 0 as end point. Hence 0 is only point for Non-uniform continuity.

Note:- Whenever we say $x=a$ point of Non-uniform continuity, it means point $x=a$ is Responsible to fail U.continuity on given Domain. Because U.continuity is Not point base Property it is global Property, which always discuss on Interval I.

Result:- Let $I=[a,\infty)$, $f: I \longrightarrow \mathbb{R}$ and function is cts on A and $\lim_{x\to\infty} f(x)$ exist then function is U.C.

Result:- Let $I=(-\infty,a]$, $f: I \rightarrow \mathbb{R}$ and function is cts on A and $\lim_{x\to-\infty} f(x)=$ exist then function is U.C.

Note:- But Reverse is Not true above two Result.

Note:- function is U.C But limit Not Need exist.

Exs:- $\sin x$

$f(x)=\sin x$ function is U.C in $\mathbb{R}$ But limit Not Exist.

Note:- $F(x) = \sin x^p$ is Not U.C in $\mathbb{R}$ $p>1$.

Ex:- $f(x) = \sin\frac{1}{x}$; $x \neq 0$, Prove U.C or Not in $(0,1]$.

he choose two point x and y in A, let $\epsilon = \frac{1}{2}$

$$x = \frac{1}{n\pi}, \quad y = \frac{1}{(2n+1)\pi/2}$$

$$|f(x) - f(y)| = \left|\sin n\pi - \sin(2n+1)\frac{\pi}{2}\right| = 1 > \epsilon$$

$$\left|\frac{1}{n\pi} - \frac{1}{(2n+1)\pi/2}\right| = \left|\frac{1}{n\pi} - \frac{2}{(2n+1)\pi}\right|$$

$$= \left|\frac{(2n+1)\pi - 2n\pi}{n\pi(2n+1)}\right|$$

$$= \left|\frac{\pi}{n\pi(2n+1)}\right|$$

$$= \left|\frac{1}{n(2n+1)\pi}\right| < \delta.$$

$\Rightarrow$ $f(x)$ is Not U.C on $(0,1]$.

Que: $f(x) = \sin x^2$; $x \neq 0$ check U.C or Not in $[0,\infty)$

het choose two point x and y in A, let $\epsilon = \frac{1}{2}$

$x = \sqrt{n\pi}$

$y = \sqrt{(2n+1)\frac{\pi}{2}}$

$|f(x)-f(y)| = \left|\sin n\pi - \sin(2n+1)\frac{\pi}{2}\right| \not< \epsilon$

$|x-y| = \left|\sqrt{n\pi} - \sqrt{(2n+1)\pi/2}\right|$

$= \left|\frac{n\pi - (2n+1)\pi/2}{\sqrt{n\pi} + \sqrt{(2n+1)\pi/2}}\right|$

$= \left|\frac{n\pi - n\pi - \pi/2}{\sqrt{n\pi} + \sqrt{(2n+1)\pi/2}}\right|$

$= \left|\frac{\pi}{2\sqrt{n\pi} + \sqrt{(2n+1)\pi/2}}\right|$

Now

$\left|\sqrt{n\pi} + \sqrt{(2n+1)\frac{\pi}{2}}\right| > \sqrt{n\pi} + \sqrt{n\pi}$

$\left|\frac{1}{\sqrt{n\pi} + \sqrt{(2n+1)\frac{\pi}{2}}}\right| < \frac{1}{\sqrt{n\pi} + \sqrt{n\pi}} = \frac{1}{2\sqrt{n\pi}}$

$< \frac{\pi}{4\sqrt{n\pi}} = \frac{\sqrt{\pi}}{4\sqrt{n}} < \delta$

$\Rightarrow$ function is Not U.C.

$\#$ $f(x) = \sin x^2$ is u.c on $[0, b]$ b be any +ve integer.

$f(x)$ is cts on $[0, b]$ $\Rightarrow$ u.c.

$\#$ $f(x) = \sin \frac{1}{x}$ is u.c on $[a, \infty)$; $a > 0$

$f(x)$ is cts in $[a, \infty)$ $a > 0$ and

$\lim_{x \to \infty} f(x) =$ Exist.

$\Rightarrow$ f is u.c.

Ques $f(x) = x^2$ on $[0, \infty)$

$x = \sqrt{n}$, $y = \sqrt{n+1}$

$|f(x) - f(y)| = |n - n - 1| = 1 > \epsilon$

But $|\sqrt{n} - \sqrt{n+1}| = \left|\frac{n-(n+1)}{\sqrt{n}+\sqrt{n+1}}\right| = \left|\frac{1}{\sqrt{n}+\sqrt{n+1}}\right| < \frac{1}{\sqrt{n}+\sqrt{n}}$

$= \left|\frac{1}{2\sqrt{n}}\right| = \frac{1}{2\sqrt{n}} < \delta$

$\Rightarrow$ function is Not u.c

But function is u.c for every bdd interval.

Q4 $f(x) = \frac{1}{x^2}$ in $(0, 1)$ u.c or Not.

$f(x)$ is cts $(0, 1)$

and $\lim_{x \to 0} f(x) =$ Not exist.

$\Rightarrow$ f is Not u.c.

Ans:

$x = \frac{1}{\sqrt{n}}, \quad y = \frac{1}{\sqrt{n+1}}$

$\left| f(x) - f(y) \right| = |n - n - 1| = 1 > \epsilon$

But $|x-y| = \left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right| = \left| \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}\sqrt{n+1}} \right|$

$= \frac{1}{\sqrt{n}\sqrt{n+1}\left(\sqrt{n+1} + \sqrt{n}\right)} < \delta$

=> Not U.C.

But $f(x) = \frac{1}{x^2}$ is U.C in $[a, \infty)$; $a > 0$

$f(x)$ is cts $[a, \infty)$ at $a > 0$ and $\lim_{x \to \infty} f(x) =$ exist
=> $f(x)$ is U.C.

Results

1) $f(x) = x$ U.C on $\mathbb{R}$
2) $f(x) = x^2, x^3$ on $\mathbb{R}$ Not U.C on $\mathbb{R}$.
3) $f(x) = \sin\frac{1}{x^2}$ is U.C $[a, \infty)$ if $a > 0$.

and Not U.C in any open interval containing zero.

4) $f(x) = \frac{1}{x}$ is U.C $[a, \infty)$ if $a > 0$

But Not U.C in any open interval containing zero and interval is subset of +ve Real Number.

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#$f(x) = e^{1/x}$ is U.C on $[a,\infty)$ if $a>0$. But Not U.C in $(0,\infty)$

#$\cos x$ is U.C in IR.

#$\cos x^2$ is Not U.C in $[0,\infty)$

#$\cos x^2$ is U.C in any bdd interval.

#$x\sin x$ is Not U.C in $[0,\infty)$

#$(x)^{1/2}, (x)^{1/3}$ is U.C on $[0,\infty)$.

See $x\sin\frac{1}{x}$ in $(0,1)$.

function is cts.

and $\lim_{x\to 0^+} x\sin\frac{1}{x} = 0$ exist

$\Rightarrow$ function is U.C.

Conditions:-

1> f is U.C on A and g is U.C on A. then f+g is U.C on A.

2> f is U.C on bdd interval, g is U.C on bdd interval then fg is U.C on bdd interval.

Note f is U.C and g is U.C then not Needed fg is U.C.

E.g:- $f(x) = x = g(x)$ is U.C on IR.

But $f(x)\cdot g(x) = x^2$ is Not U.C on IR

3) f is u.c and g is u.c then $f \circ g(x)$ or $g \circ f(x)$ is also u.c.

4) f is U.C on A and $|f(x)| > k$ for $x \in A$ where k is +ve Number. Then $\frac{1}{f(x)}$ is U.C.

5) $f : \mathbb{R} \to \mathbb{R}$, function is cts then function is U.C on Every Bdd interval.

6) $f : \mathbb{R} \to \mathbb{R}$, function is cts and Periodic, Then function is u.c.

7) function is cts on A and derivative of function is Bdd. Then function is u.c.

8) Image of open set under cts map Need Not be open.

Ex:- $f : (-1,1) \xrightarrow{cts} \mathbb{R}$

$f(x) = x^2$

$f(I) = [0,1)$ Not open

9) Uniform continuity doesn't imply Boundness.

Eg. $f(x) = x$ is u.c on $\mathbb{R}$ But Not Bdd.

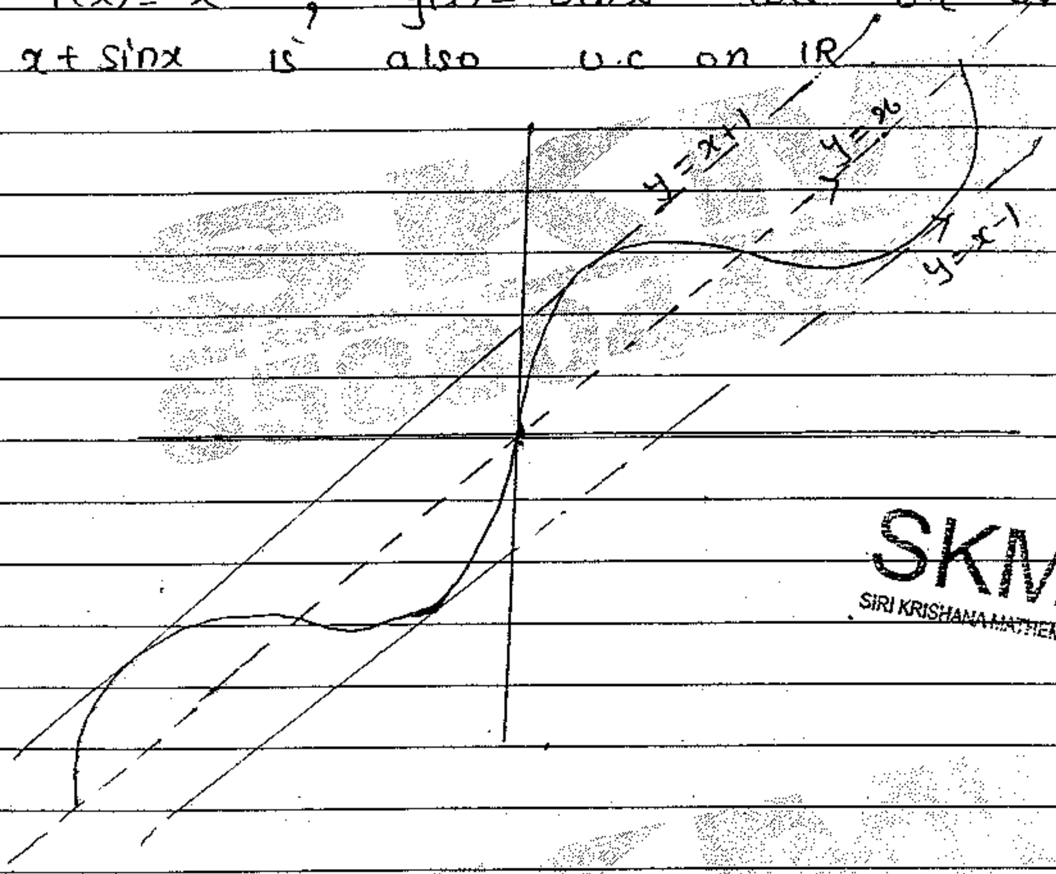
10) Rational / Division of two functions may Not be U.C.

Eg:- $f(x) = x$ and $g(x) = x^2$ are u.c on $(0,1)$

but $\frac{f(x)}{g(x)} = \frac{1}{x}$ is Not u.c on $(0,1)$.

11> If f and g are two U.C functions on I then linear combination $\alpha f + \beta g$ is also U.C on I $\forall$ $\alpha, \beta \in \mathbb{R}$.

Eg:- $f(x) = x$, $g(x) = \sin x$ are U.C on $\mathbb{R}$ then $x + \sin x$ is also U.C on $\mathbb{R}$.



Graph of $x + \sin x$ lies b/w Lines $x+1$ and $x-1$ and both lines approaches uniformely to ∞ as $x \to \infty$. Hence $f(x)$ is U.C on $\mathbb{R}$.

12> Every continuous Periodic function on $\mathbb{R}$ is always is bdd and always U.C on $\mathbb{R}$.

Fixed point Th^m :-

Let $I = [a,b]$ closed and bdd, $f : I \xrightarrow{cts} \mathbb{R}$ is cts. then $\exists$ a point $c \in I$ s.t $f(c) = c$.

# $f : \mathbb{R} \xrightarrow{cts.} \mathbb{R}$

$f(x) = x^2$

Here two fixed point 0, 1.

0, 1

$f(x) = x^2, \quad y = x$

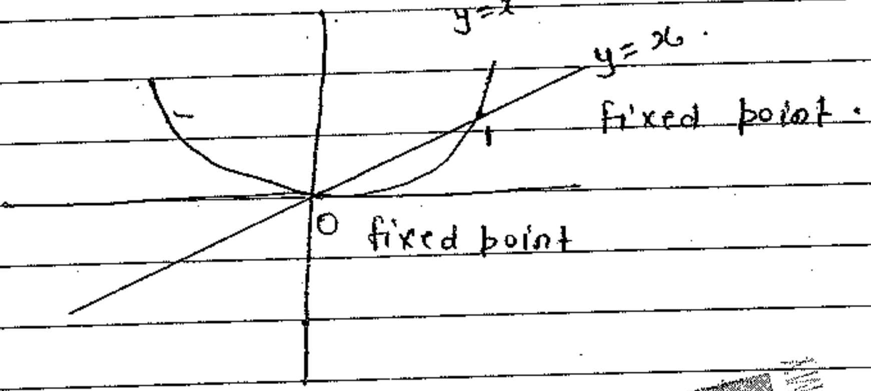
$x = x^2$

$x - x^2 = 0$

$x(x-1) = 0$

$x = 0, x = 1$

By Graphical.



जितने point पर cut करता है वो fixed point बनते हैं।

# $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$

$f(x) = \sin x$

$\sin x = x$

$\sin x - x = 0$

$x = 0$

i.e one fixed point, 0.

# $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$

$f(x) = \sin x + x$.

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$x = \sin x + x$

$\sin x = 0$

$x = n\pi$ (fixed point)

(Countably infinite).

# $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$

$f(x) = x$

unc'ble fixed point.

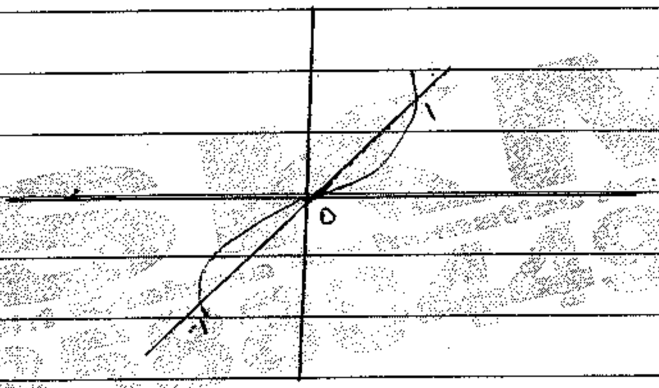
# $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$

$f(x) = x+1$

No. fixed point.

# $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$

$f(x) = x^3$ has three fixed point. $1, 0, -1$.



Qu How many function $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$ which have exactly one fixed point.

$f(x) = a \quad a \in \mathbb{R}$

or $f(x) = ax \quad a \neq 1 \quad a \in \mathbb{R}$.

i.e. unc'ble.

# 1) $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$; $S = \{x \,;\, f(x) > 0\}$ is open set

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2) $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$; $S = \{x ; f(x) < 0\}$ is open set.

3) $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$; $S = \{x ; f(x) \neq 0\}$ is open set

NET 4) $f : \mathbb{R} \xrightarrow{cts} \mathbb{R}$; $S = \{x : f(x) = 0\}$ is closed set

5) $f : \mathbb{R} \xleftrightarrow{cts} \mathbb{R}$ and $g : \mathbb{R} \xrightarrow{cts} \mathbb{R}$; $S = \{x : f(x) > g(x)\}$ is open

6) $f : \mathbb{R} \xleftrightarrow{cts} \mathbb{R}$ and $g : \mathbb{R} \xrightarrow{cts} \mathbb{R}$; $S = \{x : f(x) \neq g(x)\}$ is open.

7) $f : \mathbb{R} \xleftrightarrow{cts} \mathbb{R}$ and $g : \mathbb{R} \xrightarrow{cts} \mathbb{R}$; $S = \{x : f(x) = g(x)\}$ is closed

1) $f(x) = k$; $k > 0$.

$S = \mathbb{R}$ open.

2) Reverse.

3) $f(x) \neq 0$

$f(x) < 0$ or $f(x) > 0$

↓ ↓

open open.

$f(x) \neq 0 \Rightarrow S$ is open.

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4) $f(x) = x$

$f(0) = 0$

i.e $S = \{0\}$ is closed.

or $f(x) = \sin x$.

$\Rightarrow$ $S = \{n\pi ; n \in \mathbb{Z}\}$.

$S' = \phi$.

$\Rightarrow S$ is closed.

and other are similar.

$f : I \longrightarrow \mathbb{R}$ is uniformly continuous. $\{x_n\}$ is cauchy seq^n then $f(x_n)$ is cauchy seq^n.

Note:- if $f : I \longrightarrow \mathbb{R}$ is continuous. $\{x_n\}$ is cauchy seq^n But $f(x_n)$ is Need Not be Cauchy seq^n.

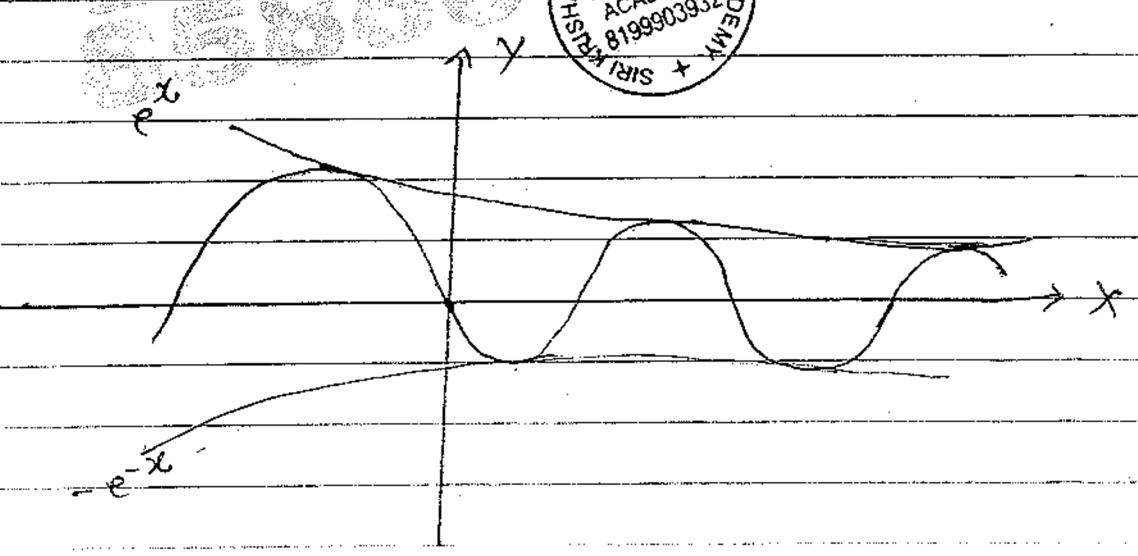
E.g:- $f(x) = \frac{1}{x}$ on $(0,1)$

Let $\langle a_n \rangle = \langle \frac{1}{n} \rangle \in (0,1)$ is cauchy But $f(a_n) = n$ is Not cauchy.
Hence $f(x) = \frac{1}{x}$ is Not u.c on $(0,1)$.

ie if $\langle a_n \rangle$ is cauchy but $f(a_n)$ is Not cauchy then f is Not u.c.

if $f : S \longrightarrow \mathbb{R}$ be cts on S then Range Set $f(S) = \{f(x),\ x \in S\}$ is closed and bounded.

Qu $f(x) = e^{-x} \sin x$.



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$f(x)$ is cts on $\mathbb{R}$ But Not U.C on $\mathbb{R}$

Because as $x \to -\infty$ $f(x)$ oscillate b/w $-\infty, \infty$,

Hence, we can say $f(x)$ is U.C on $(0, \infty)$

and on any interval of finite length.

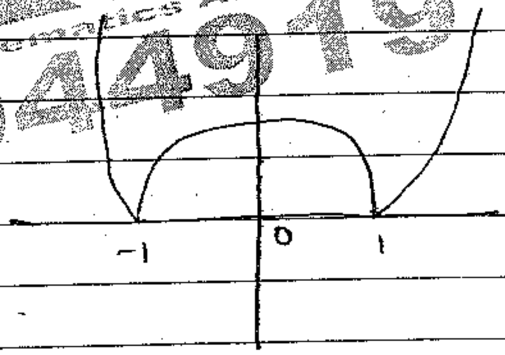
Qu $f(x) = |x^2 - 1|$

Not U.C on $\mathbb{R}$

U.C on Every

finite length of

interval.



$f : [a,b] \longrightarrow \mathbb{R}$ is U.C then function is bdd.

$f : [a,b] \longrightarrow \mathbb{R}$ is U.C then Function has supermum and infrimum.

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1) Image of compact set under U.C is compact.
2) Image of connected set under U.C is compact.
3) Image of closed and bdd set under U.C is closed & bdd.

Functions and Graphs.

Functions.

i) Algebraic.

ii) Transcedental

① **Algebraic.**

i) Polynomial

ii) Rational

iii) Irrational

iv) **Peicewise.**

Modulus | **Signum** | **Greateast Integer** | **Fractional Part** | **Least Integer**

② **Transcendental**

(i) Trigonometric

(ii) Exponential

(iii) Logarithmic / Inverse of exponential

(iv) Geometrical

(v) Inverse Trigonometric.

1> Algebraic Function :- function $y = f(x)$ is said to be Algebraic if it is Root of Polynomial in y of the form $\sum_{i=1}^{n} P_i(x)y^i = 0$, where $P_i(x)$ are Polynomial over $\mathbb{R}$.

i.e. $P_0(x) + P_1(x)y + P_2(x)y^2 + --- + P_n(x)y^n = 0$

Eg:- $(2x+1)y^2 - yx^2 = 0$

$y[y(2x+1)-x^2]=0$

$\Rightarrow y=0$ and $y=\frac{x^2}{x^2+1}$

So function $y=\frac{x^2}{x^2+1}$ is Root of Polynomial

$(2x+1)y^2-yx^2=0$ so $y=\frac{x^2}{x^2+1}$ is Algebraic.

Eg:- $(2x+1)y^2-yx^2+1=0$

$y=\frac{x^2\pm\sqrt{x^4-4(2x+1)}}{2\cdot(2x+1)}$ is algebraic function.

Eg: $y^2-(\pi x+1)^2=0$

$\Rightarrow y=\pm(\pi x+1)$ Algebraic function.

Types of Algebraic Functions:-

1) Polynomial function:- A function of the from $f(x)=a_0+a_1x+a_2x^2+\dots+a_nx^n$, $n\in\mathbb{N}$ & $a_i\in\mathbb{R}$ is called Polynomial function.

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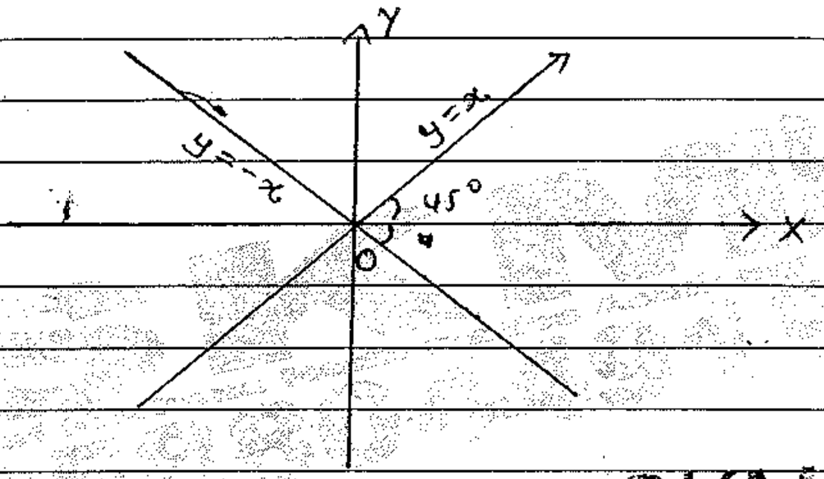
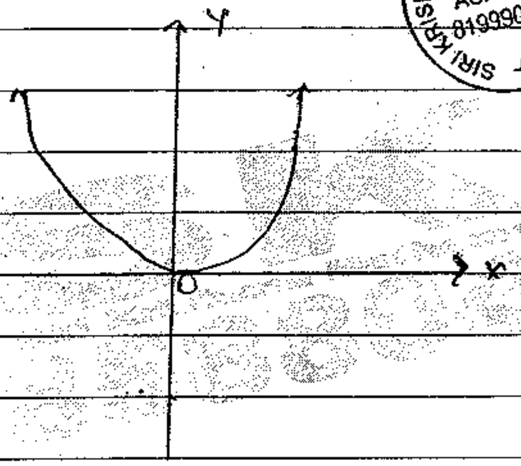
Some Polynomials Functions with graph.

1) Identity function $y = x$



Dod of $y = x = \mathbb{R}$
Range of $y = x = \mathbb{R}$.

(ii) Square function $y = x^2$



$y = x^2$ is even function. So It's graph will be Symmetric about y-axis.

Dod $= \mathbb{R}$
Range $= \mathbb{R}^+ \cup \{0\}$

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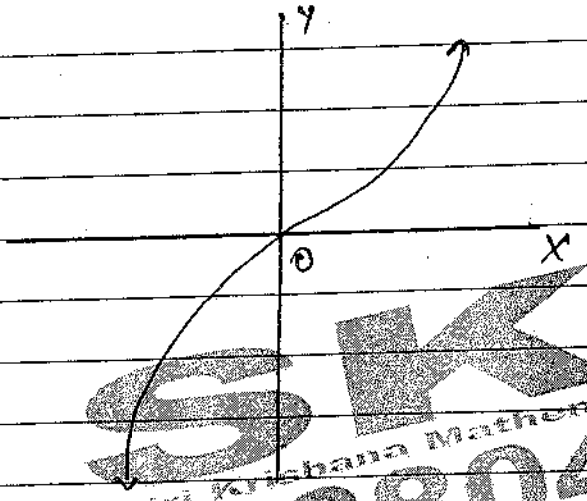
(iii) cube function $y=x^3$

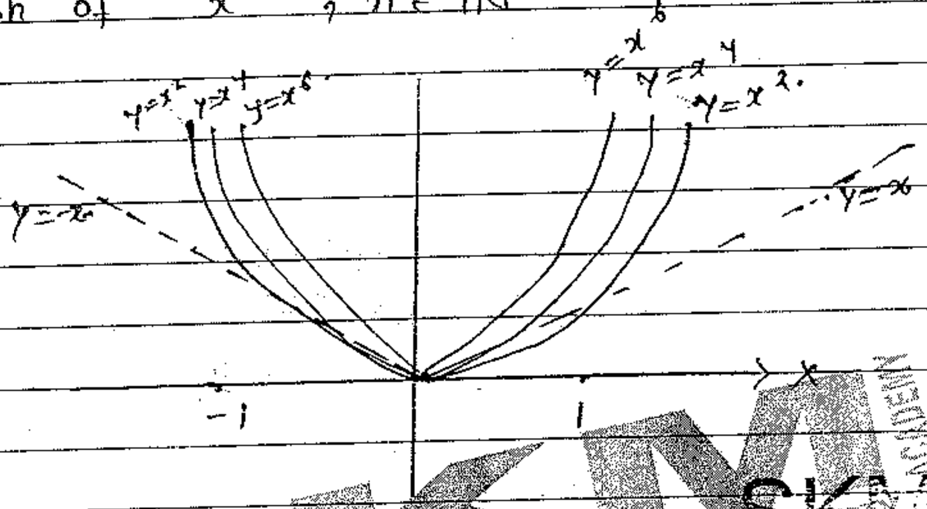


$y=x^3$ is an odd function so it's graph will be symmetric about opposite quadrant i.e symmetric about origin.

Dom = $\mathbb{R}$

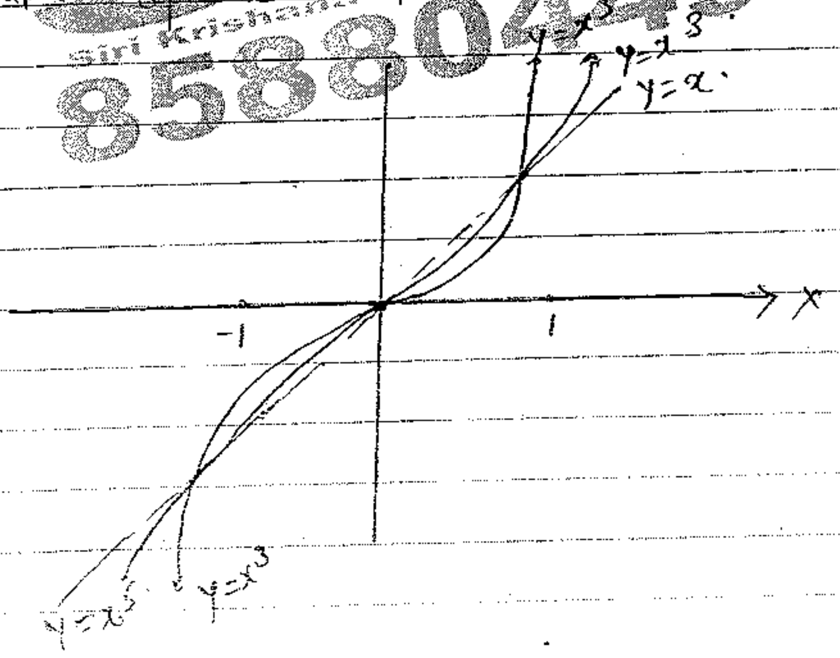
Range = $\mathbb{R}$.

(iv) Graph of x^{2n}, $n \in \mathbb{N}$



$y=x^2$
$y=x^4$
$y=x^6$
$y=x^6$
$y=x^4$
$y=x^2$
$y=-x$
$y=x$
-1
1
x

v) Graph of x^{2n-1}, $n \in \mathbb{N}$.



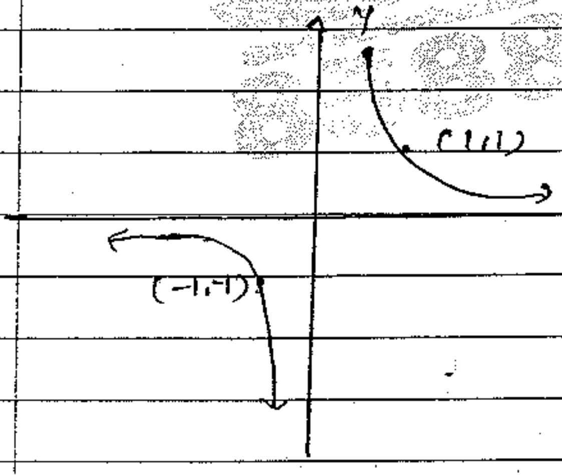
$y=x^5$
$y=x^3$
$y=x$
-1
1
x
$y=x^5$
$y=x^3$

II Rational function :- A function obtained by dividing a polynomial by another polynomial is called Rational function.

i.e $f(x) = \frac{P(x)}{Q(x)}$

Some Rational function with graph.

i) $f(x) = \frac{1}{x}$ [Reciprocal function or Rectangular Hyperbola

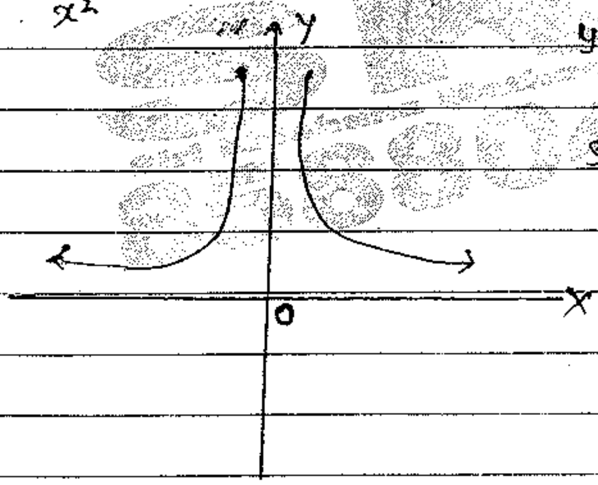


$y = \frac{1}{x}$ is an odd function so graph will be symmetric about origin.

Dod = $\mathbb{R} - \{0\}$

Range = $\mathbb{R} - \{0\}$

ii) $y = \frac{1}{x^2}$



$y = \frac{1}{x^2}$ is even function so symmetric about X-axis.

Dod = $\mathbb{R} - \{0\}$. and

Range = $(0, \infty)$.

III> Irrational Functions :- The algebraic function containg terms having Non-integral Rational Powers of x are called Irrational functions.

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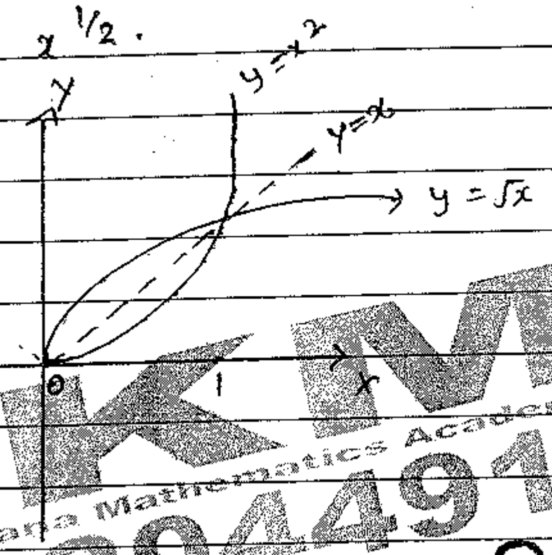
Eg:- $f(x) = \sqrt{x} = x^{1/2}$.


y
$y=x^2$
$y=x$
$y=\sqrt{x}$
0
1
x

Dod = $\mathbb{R}^+ \cup \{0\}$

Range = $\mathbb{R}^+ \cup \{0\}$.

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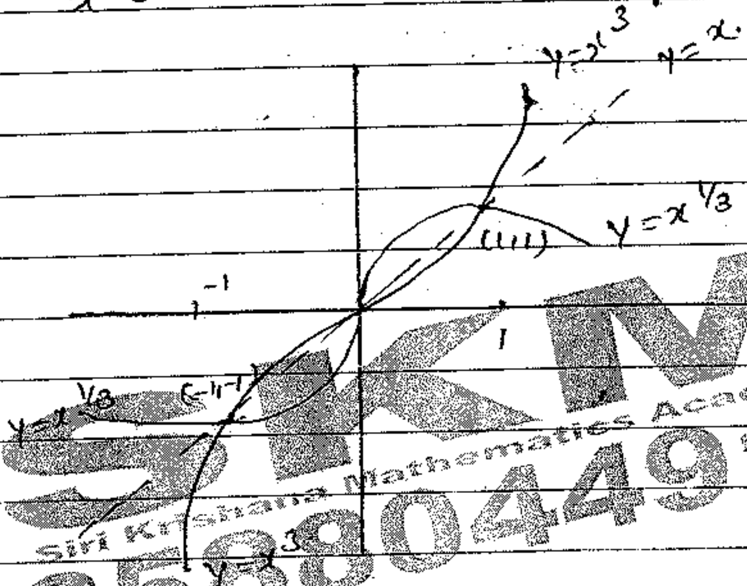
Eg:- $f(x) = x^{1/3}$.


$y=x^3$
$y=x$
$y=x^{1/3}$
(1,1)
-1
1
(-1,-1)
$y=x^{1/3}$
$y=x^3$

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Note:- $f(x)=x^n$ and $g(x) = x^{1/n}$, then $f(x)$ and $g(x)$ are inverse of each other.

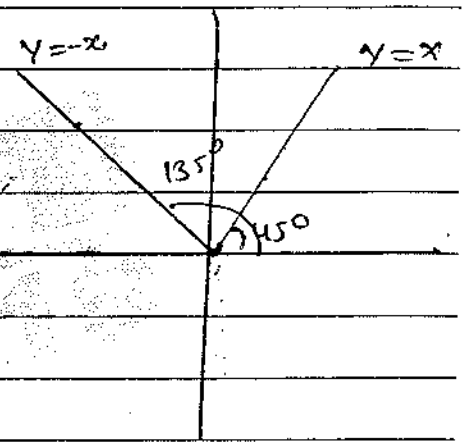
Note:- Graph of $g(x) = x^{1/n}$ is the mirror image of graph of $f(x) = x^n$ about line $y = x$.

IV) Piecewise functions:-

a) Modulus functions (or Absolute value function).

function $y = |x| = \begin{cases} x & ; x \ge 0 \\ -x & ; x < 0 \end{cases}$

is called Modulus function.



This is even function so graph Symmetoric about y- axis.

Dod = IR , Range = $[0, \infty)$.

Properties of Modulus Function :-

i) $|x| \le a \Rightarrow -a \le x \le a$; $(a \ge 0)$

ii) $|x| \ge a \Rightarrow x \le -a$ or $x \ge a$; $(a \ge 0)$

iii) $|x \pm y| \le |x| + |y|$

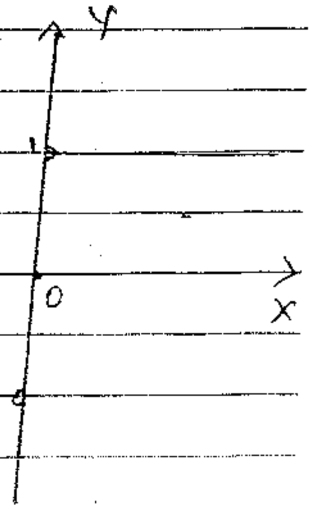
iv) $|x \pm y| \ge \big||x| - |y|\big|$

b) Signum function [y = Sig(x)].

It is defined by.

$$y = \operatorname{Sig}(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|} & ; x \ne 0 \\ 0 & ; x = 0. \end{cases}$$

$$= \begin{cases} +1 & ; x > 0 \\ -1 & ; x < 0 \\ 0 & ; x = 0 \end{cases}$$



Dod = IR ; Range = $\{-1, 0, 1\}$.

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c) Greatest Integer function [$y = [x]$].

function $y = [x]$ indicates the integral part of x which is Nearest and smallest integer to x. known as 'floor of x'. or Stepwise function.

Eg:- $[2.7]$ = Nearest & smallest Integer to 2.7.

$= 2$.

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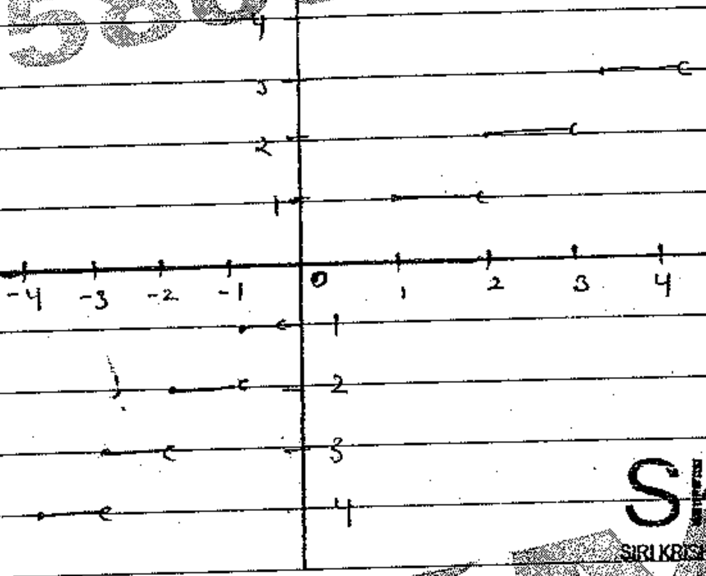

(Eg) $[-7.02] = -8$.

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In General

$n \leq x < n+1 \ (n \in Z)$.

$\Rightarrow [x] = n$.

Dod $= \mathbb{R}$

Range $= Z$.

Properties :-

1) $[x] = x$ if $x \in Z$ [Integer].

ii) $[x+y] \ge [x]+[y]$.

iii) if $[\phi(x)] \ge I$ then $\phi(x) \ge I$, Here $I \in Z$.

iv) if $[\phi(x)] \le I$ then $\phi(x) < I+1$.

v) $[-x] = -[x]$; $x \in Z$.

vi) $[-x] = -[x]-1$ if $x \notin Z$.

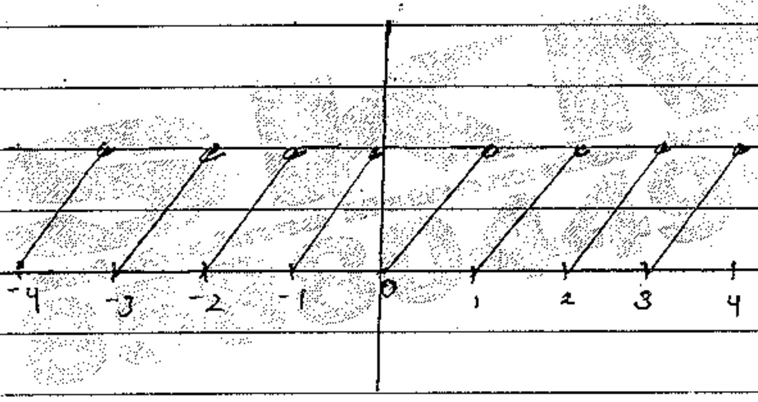
d) Fractional Part of Function $y = \{x\}$.

Here $\{\cdot\}$ denotes the fractional part of x.
Thus we can find fractional part of x like as.

$\{x\} = x-[x]$ i.e. $y = \{x\} = x-[x]$.

Eg:- $\{3.021\} = 3.021 - [3.021] = 3.0.21 - 3.0 = .021$.
So $.021$ is fractional part of 3.021.

Note:- From the defination of $\{x\}$, we have a conclusion that $\{x\} \in [0,1)$ $\forall$ $x \in \mathbb{R}$



Dod $= \mathbb{R}$.
Range $= [0,1)$.

Properties :-

i) $\{x\} = x$ if $0 \le x < 1$

ii) $\{x\} = 0$ if $x \in$ Integer.

iii) $-\{x\} = 1-\{x\}$; $x \in$ Integer.

(e) Least Integer function [$y = (x)$, $\lceil x \rceil$]

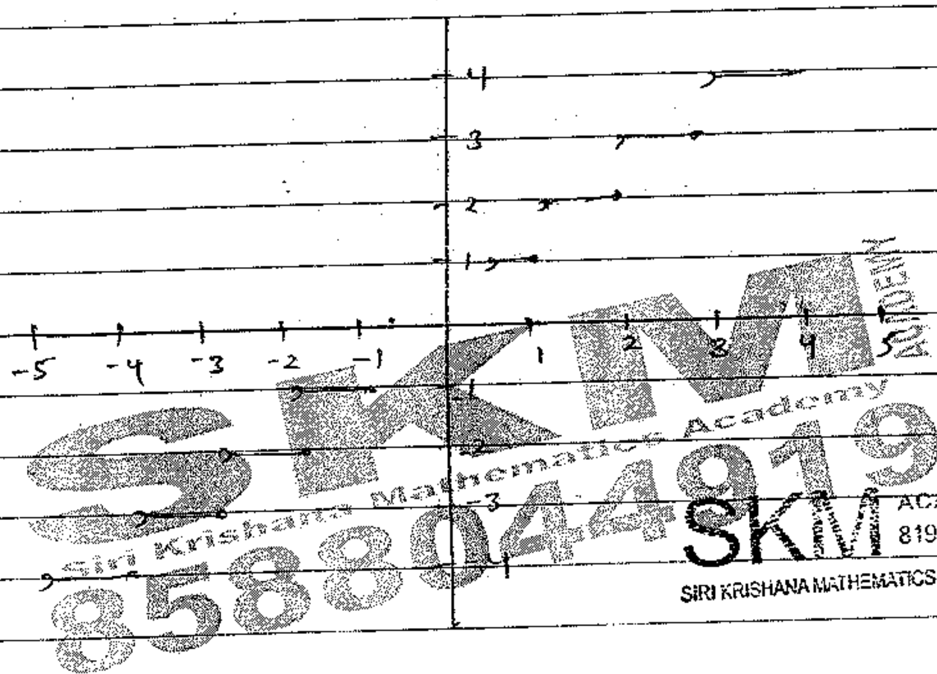
1) $y=(x)$ or $[x]$ indicates the integral part of x which is Nearest and greatest integer to x, known also as ceilling of x.

Eg:- $[2.1301]$ = Nearest and largest integer to 2.1301
$= 3$

$[-10.13] = -10$.

Properties:- i) $\lceil x \rceil = x$; if x is integer.

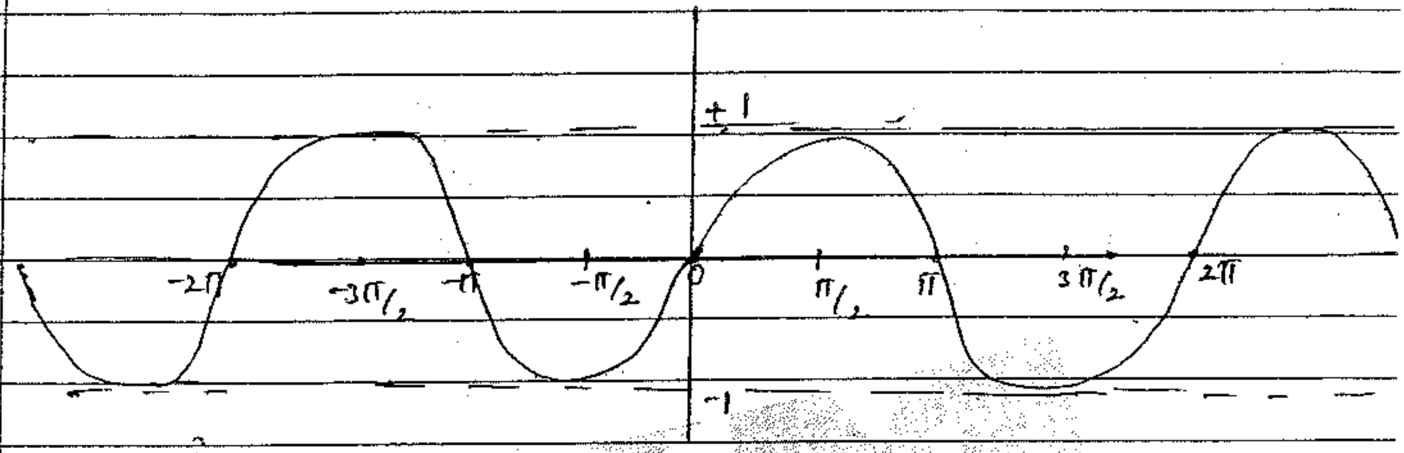
ii) $\lceil x+I \rceil = \lceil x \rceil + I$; if $I \in$ Integer.



② Transcendental functions :-

1) Trigonometrical functions :-

i) Sine function, [$y = \sin x$].



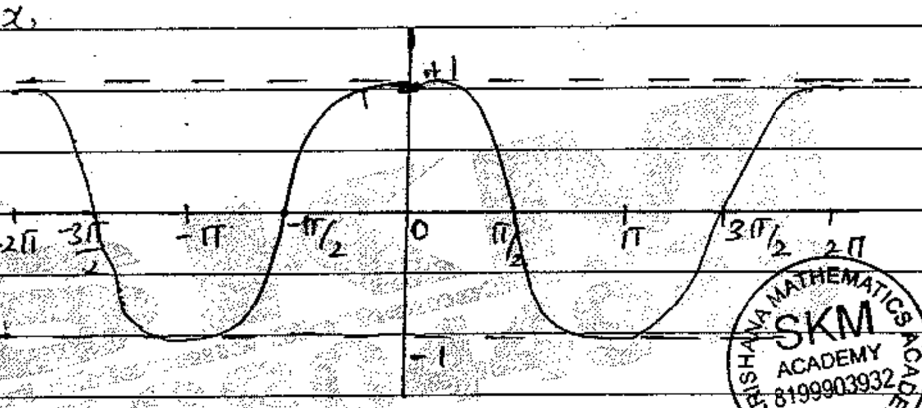
$\sin(n\pi) = 0 \quad \forall \ n \in \mathbb{N}$

$\sin(4n-3)\frac{\pi}{2} = 1 \quad \forall \ n \in \mathbb{N}$

$\sin(4n-1)\frac{\pi}{2} = -1 \quad \forall \ n \in \mathbb{N}$

Dod = $\mathbb{R}$, Range = $[-1, 1]$.

b) $y = \cos x$.



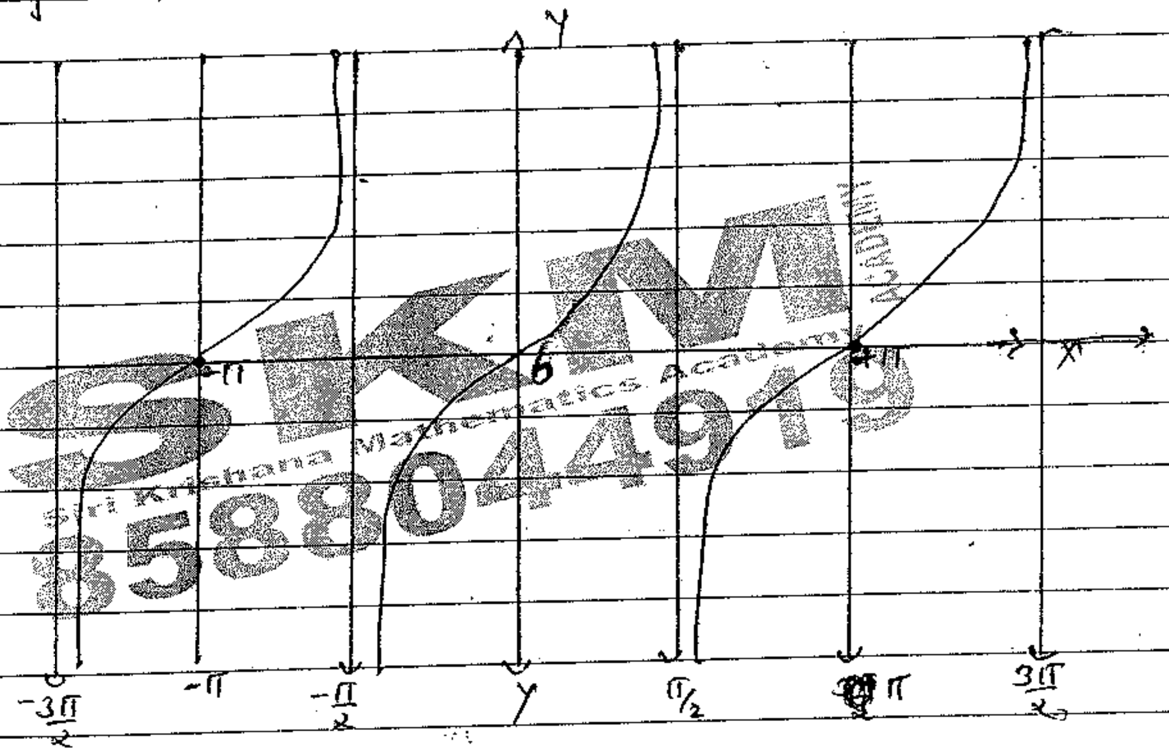
$\cos(2n\pi) = 1$; $n \in \mathbb{Z}$.

$\cos(2n+1)\pi = -1$

$\cos(2n+1)\frac{\pi}{2} = 0$

Dod = $\mathbb{R}$, Range = $[-1, 1]$.

c) tangent function [y = tan x].

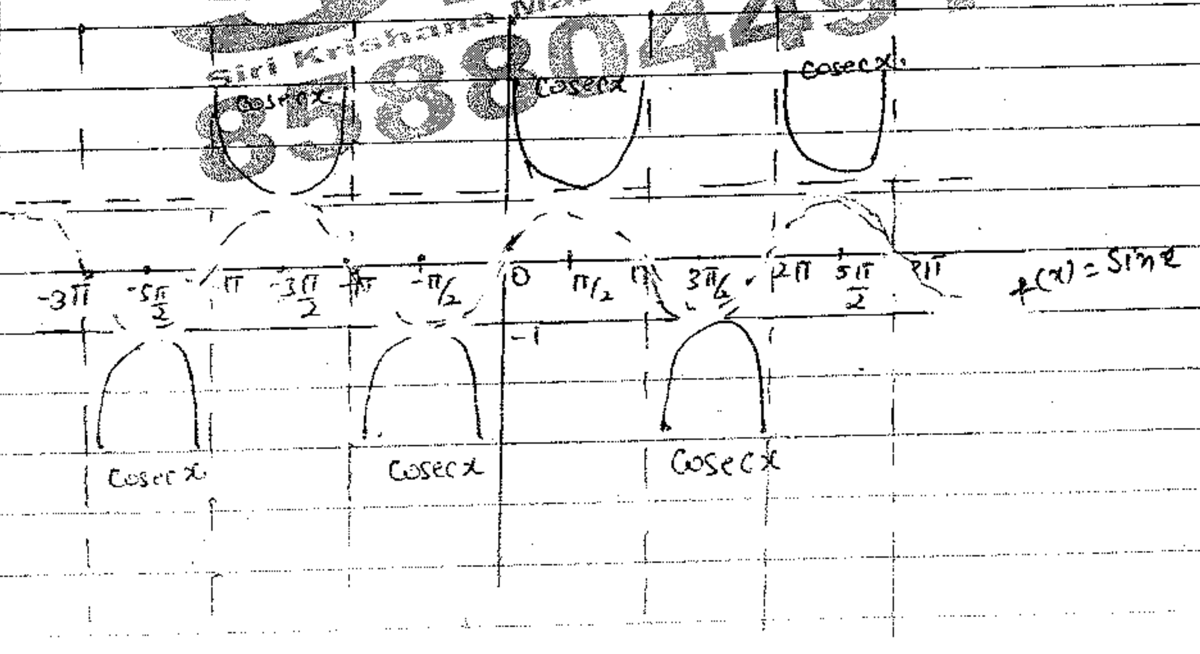


$\tan\left[\pm(2n-1)\frac{\pi}{2}\right] = \infty$

So DOD = $\mathbb{R} - \left[\pm(2n-1)\frac{\pi}{2}\right]$ $n \in \mathbb{N}$

Range = $(-\infty, \infty)$ or $\mathbb{R}$.

d) Cosecant function [y = cosec x]



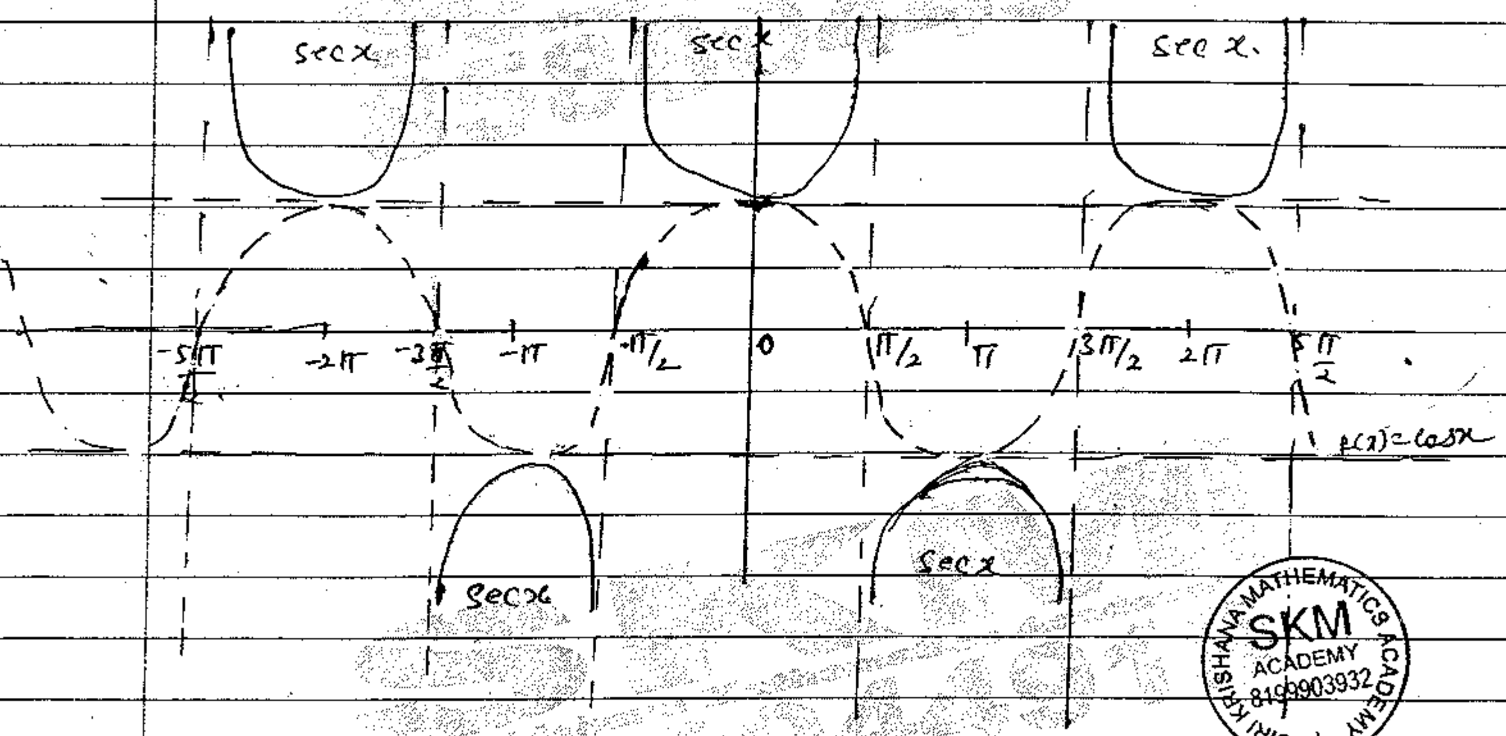
$y = \operatorname{cosec} x = \frac{1}{\sin x}$

So cosecx is Not defined at ± nπ so

DoD = $\mathbb{R} - \{n\pi ; n \in \mathbb{Z}\}$.

Range = $\mathbb{R} - (-1, 1)$.

(e) Secant function [$y = \sec x$].



$y = \sec x = \frac{1}{\cos x}$

So DoD = $\mathbb{R} - \{(2n+1)\frac{\pi}{2} ; n \in \mathbb{Z}\}$.

Range = $\mathbb{R} - (-1, 1)$. → ?

(f) Cotangent function [$y = \cot x$]

DoD = $\mathbb{R} - \{n\pi ; n \in \mathbb{Z}\}$

Range = $\mathbb{R}$.

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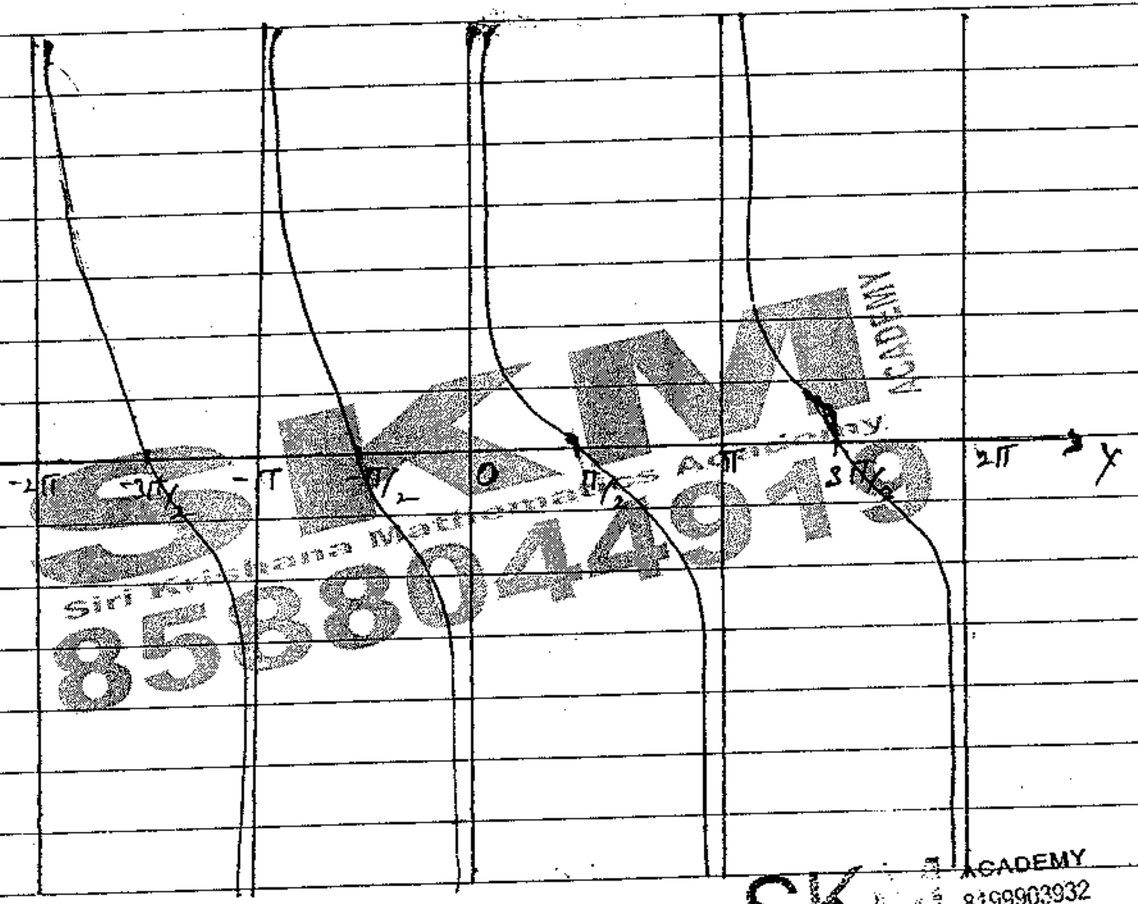

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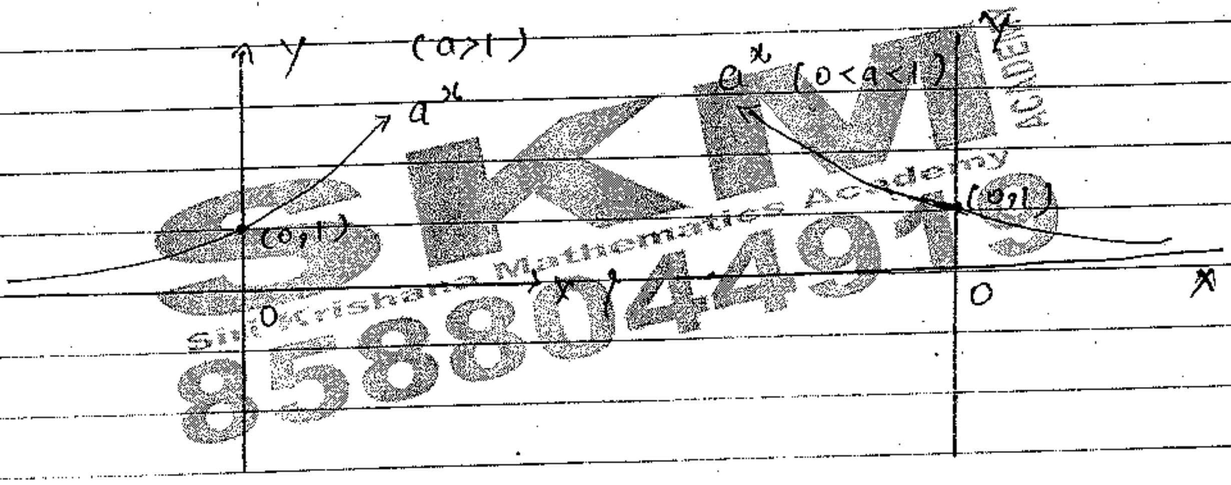
II Exponential function. $[f(x) = a^x,\ a>0\ \&\ a\neq 1]$.



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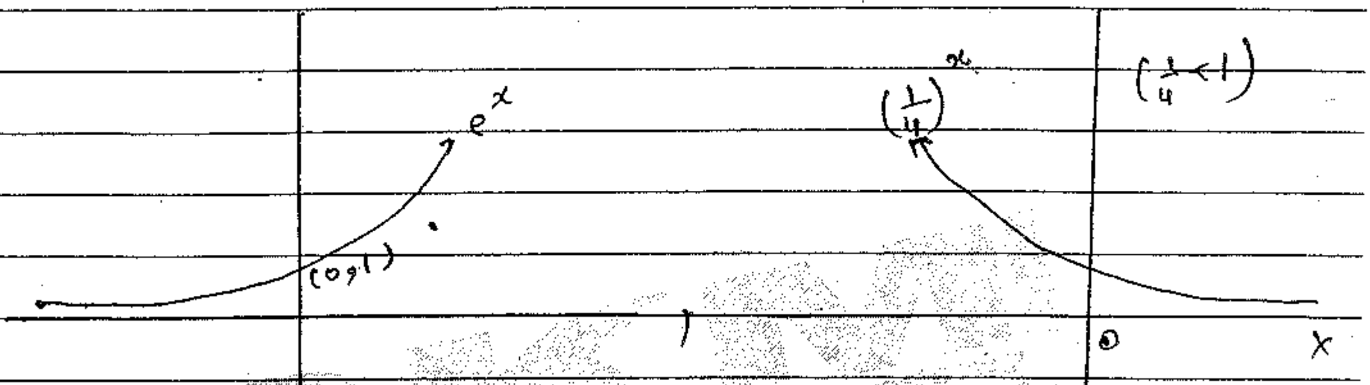
Dod = $\mathbb{R}$

Range = $(0,\infty)$.

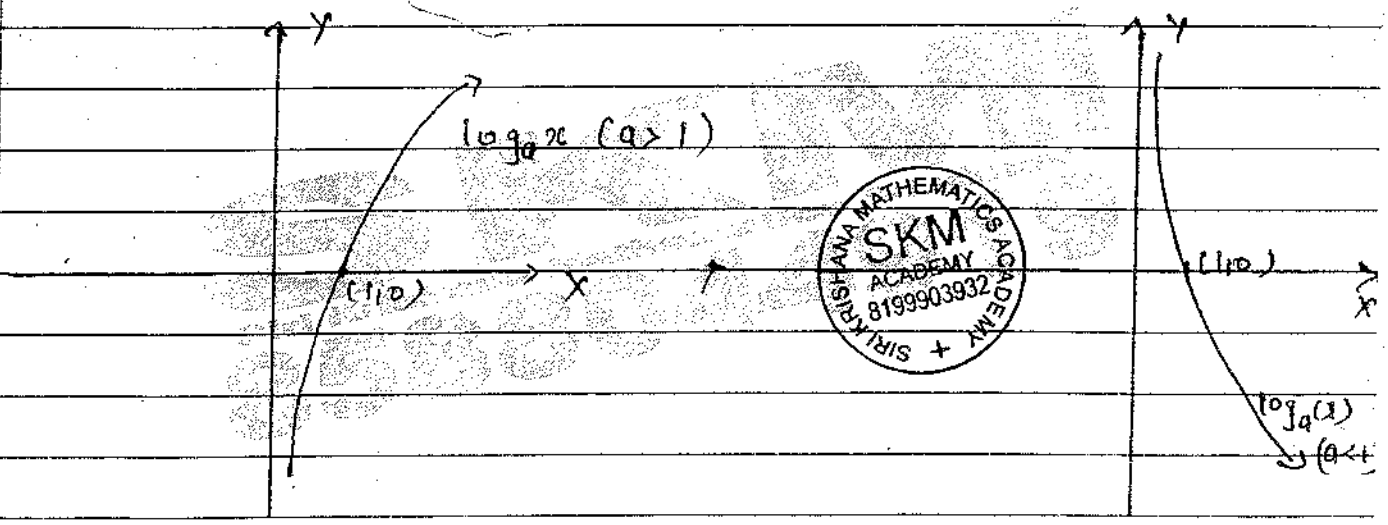
Exponential function increasing and decreasing when $a>1$ and $0<a<1$ Respectively.

Eg:- e^x $(e>1)$



III Logarithmic Function $[\,y = \log_a x,\ x \,\&\, a>0\,]$

Logarithmic function always defined for +ve Real Number so dod = IR. $y = \log_a x$ is inverse of exponential function of a^x. Range = IR.



Properties of logarithmic function :-

(1) $\log_e(a\cdot b) = \log_e a + \log_e b$ $[a,b>0]$

② $\log_e(\frac{a}{b}) = \log_e a - \log_e b$ $[a,b>0]$

③ $\log_e a^m = m\log_e a$ $[a>0 \,\&\, m\in\mathbb{R}]$

④ $\log_a a = 1$ $[a>0 \ \& \ a\neq 1]$.

⑤ $\log_{b^m} a = \frac{1}{m}\log_b a$ $[a,b>0, \ b\neq 1 \ \& \ m\in\mathbb{R}]$.

⑥ $\log_b a = \frac{1}{\log_a b}$ $[a,b>0 \ \& \ a,b\neq 1]$.

⑦ $\log_b a = \frac{\log_m a}{\log_m b}$ $[a,b>0 \ \& \ a,b\neq 1 \text{ and } m>0]$

⑧ $a^{\log_a m} = m$ $[a,m>0 \text{ and } a\neq 1]$.

⑨ $a^{\log_c b} = b^{\log_c a}$ $[a,b,c>0 \text{ and } c\neq 1]$.

⑩ if $\log_m x > \log_m y \Rightarrow \begin{cases} x>y & \text{if } m>1 \\ x<y & \text{if } 0<m<1 \end{cases}$

$[m, x, y>0 \text{ and } m\neq 1]$

⑪ $\log_m a = b \Rightarrow a = m^b$ $[a, m>0, \ m\neq 1, \ b\in\mathbb{R}]$

⑫ $\log_m a > b \Rightarrow \begin{cases} a>m^b & \text{if } m>1 \\ a<m^b & \text{if } 0<m<1 \end{cases}$

⑬ $\log_m a < b \Rightarrow \begin{cases} a<m^b & \text{if } m>1 \\ a>m^b & \text{if } 0<m<1 \end{cases}$

IV **Inverse Trigonometric function :-**

Trigonometric functions are many one functions in their Domain, Hence they are Not invertible.

But their Inverse can be obtained by Restricting the Domain so as to make invertible (bijective).

i.e. Every inverse Trigonometric is been converted to a function by Shortening the Domain.

for Example :- Let $f(x) = \sin x$ [Not invertible].
but $\sin x$ is one~~and~~onto in Domain $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ so,

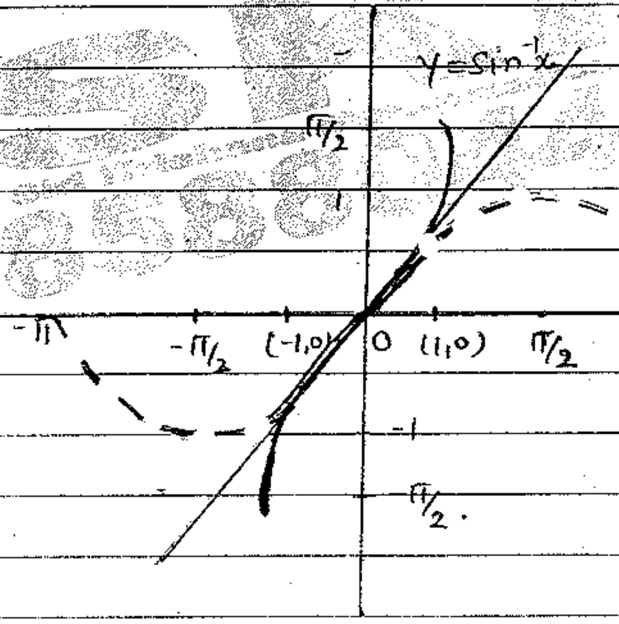
if $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ s.t

$f(x) = \sin x$ then $f(x)$ is invertible,
and then the inverse of $f(x)$ can be defined by

$$f : [-1, 1] \longrightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$f(x) = \sin^{-1} x..$$

Note :- The graph of inverse function of given function is always mirror image of given function about line $y = x$.



we are taking mirror image of $\sin x$ for Domain $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, because in this Domain

$f(x) = \sin x$ is invertible. we define inverse then Domain and co-domain are changed.

Similarily.

(2) $y = \cos x$, becomes invertible when.

$$f : [0, \pi] \longrightarrow [-1, 1].$$

(3) $y = \tan x$, becomes invertible when.

$$f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \longrightarrow (-\infty, \infty).$$

(4) $y = \cot x$, becomes invertible when

$$f : (0, \pi) \longrightarrow (-\infty, \infty).$$

(5) $y = \sec x$, becomes invertible when

$$f : [0, \pi] - \frac{\pi}{2} \longrightarrow \mathbb{R} - (-1, 1)$$

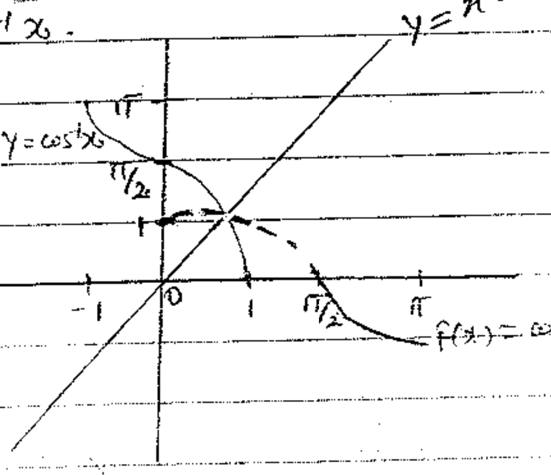
(6) $y = \operatorname{cosec} x$, becomes invertible when

$$f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\} \longrightarrow \mathbb{R} - (-1, 1).$$

Graph of $y = \cos^{-1} x$.

$f : [0, \pi] \longrightarrow [-1, 1]$.

$f(x) = \cos^{-1} x$.



Here Red Portion Shows the mirror image of graph of $f(x) = \cos x$.

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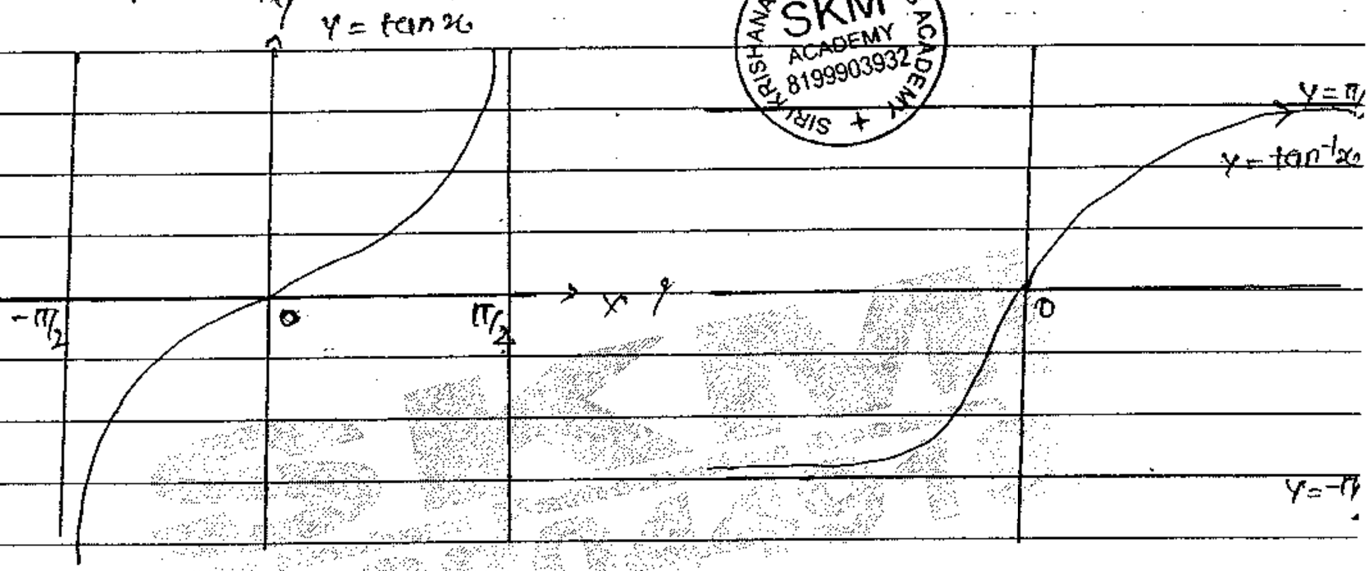
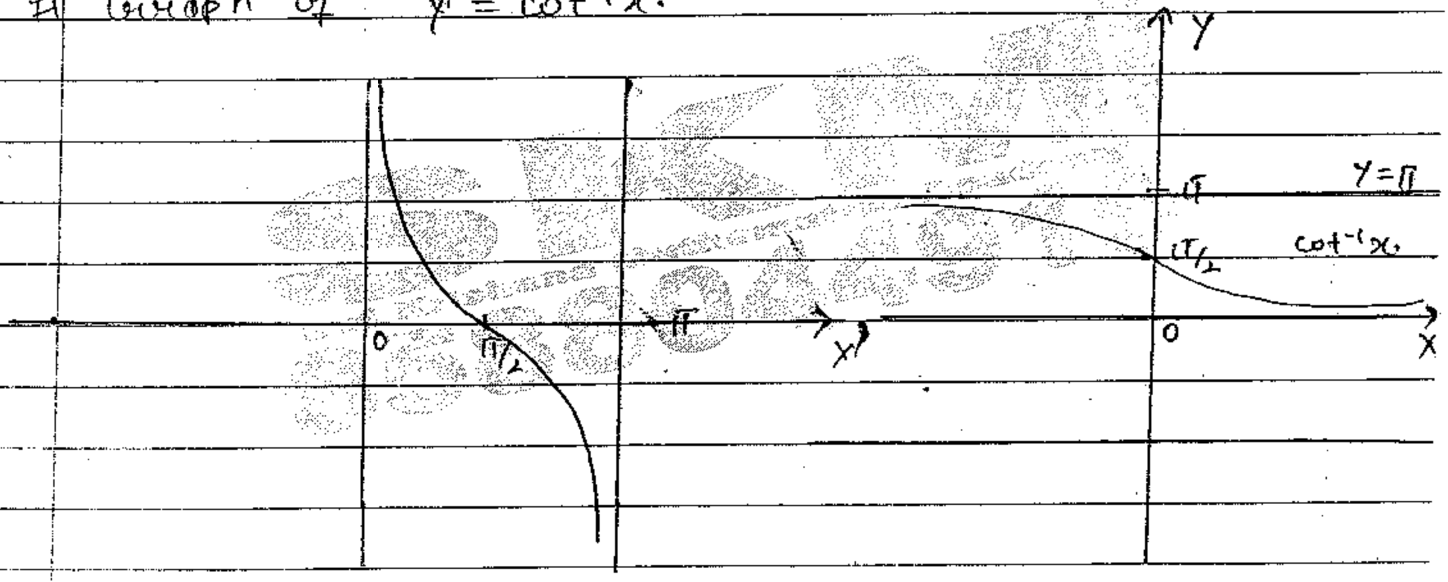
Graph of $\tan^{-1} x$.



Note:- Graph of inverse ~~a~~ function is image of $f(x)$ about $y = x$ or by interchanging the coordinate axis.

Graph of $y = \cot^{-1} x$.

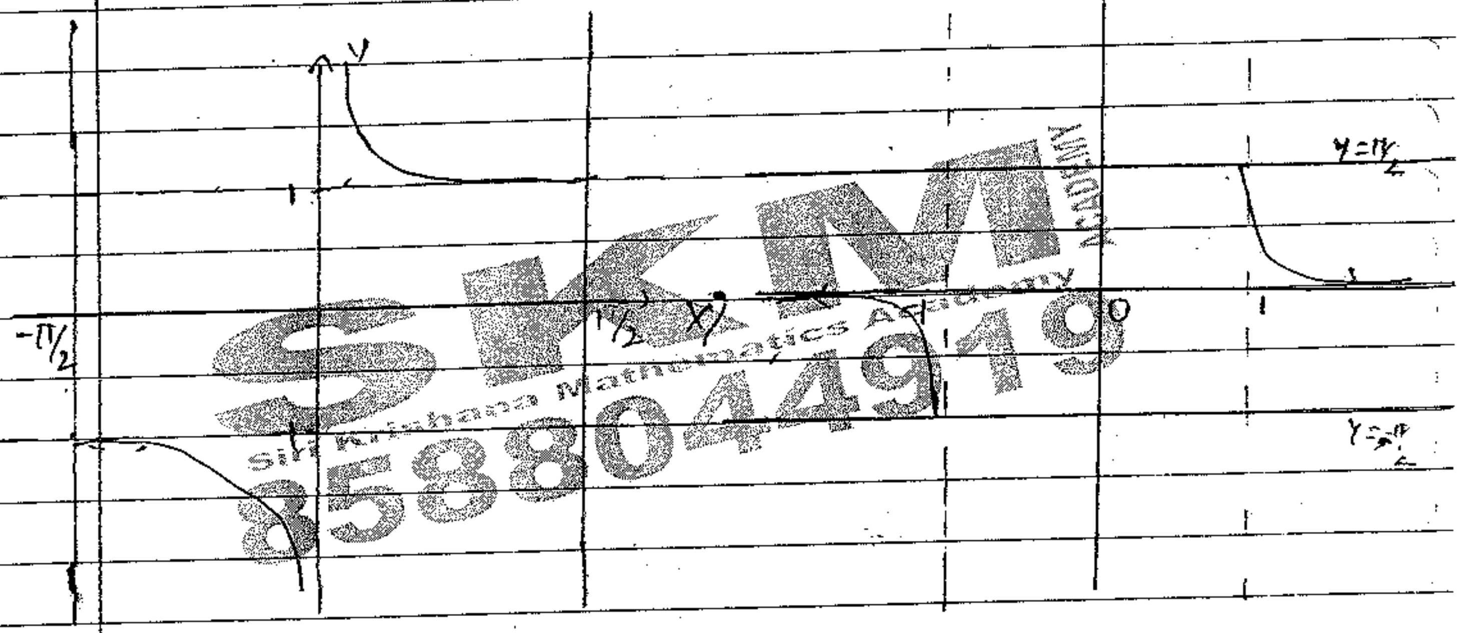


$f : (0, \pi) \longrightarrow \mathbb{R}$.

$f(x) = \cos^{-1} x$.

Graph of ~~$\cos^{-1}x$~~, $\operatorname{cosec}^{-1} x$

$f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\} \longrightarrow \mathbb{R} - (-1,1)$.



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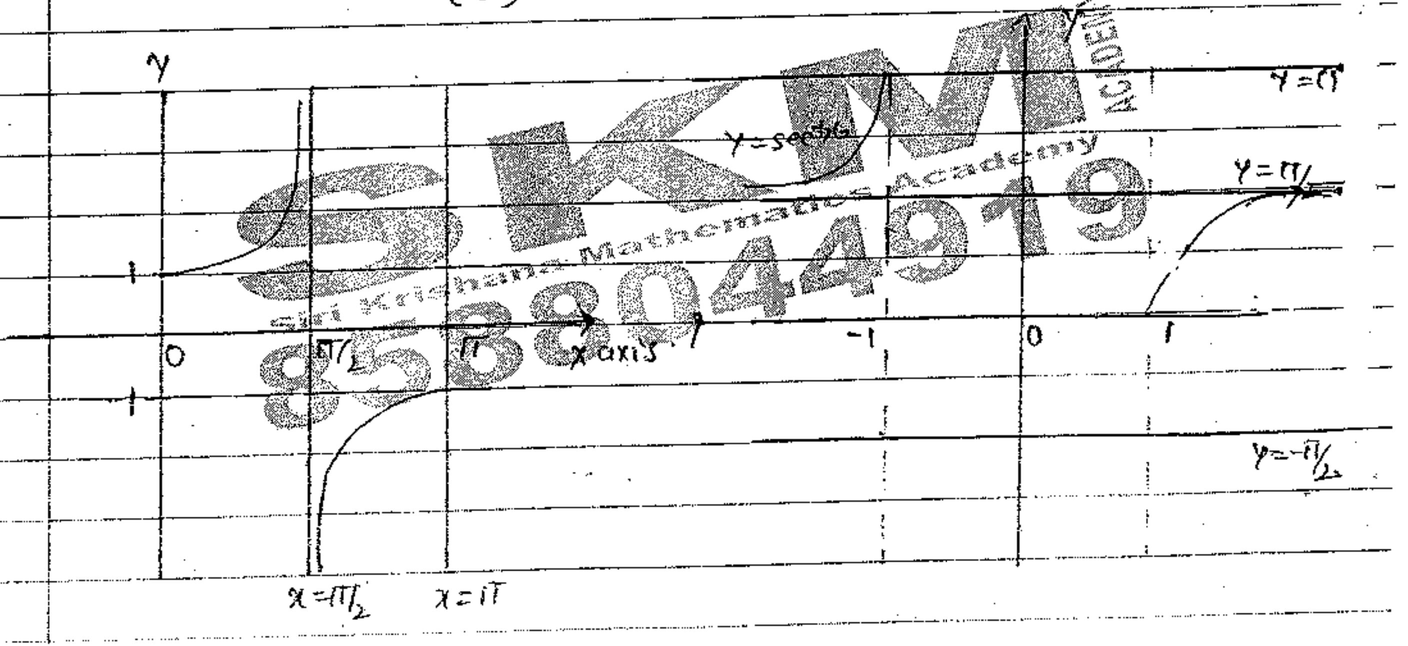
Graph for $y = \sec^{-1}x$.

$f : [0, \pi] - \left\{\frac{\pi}{2}\right\} \longrightarrow \mathbb{R} - (-1,1)$



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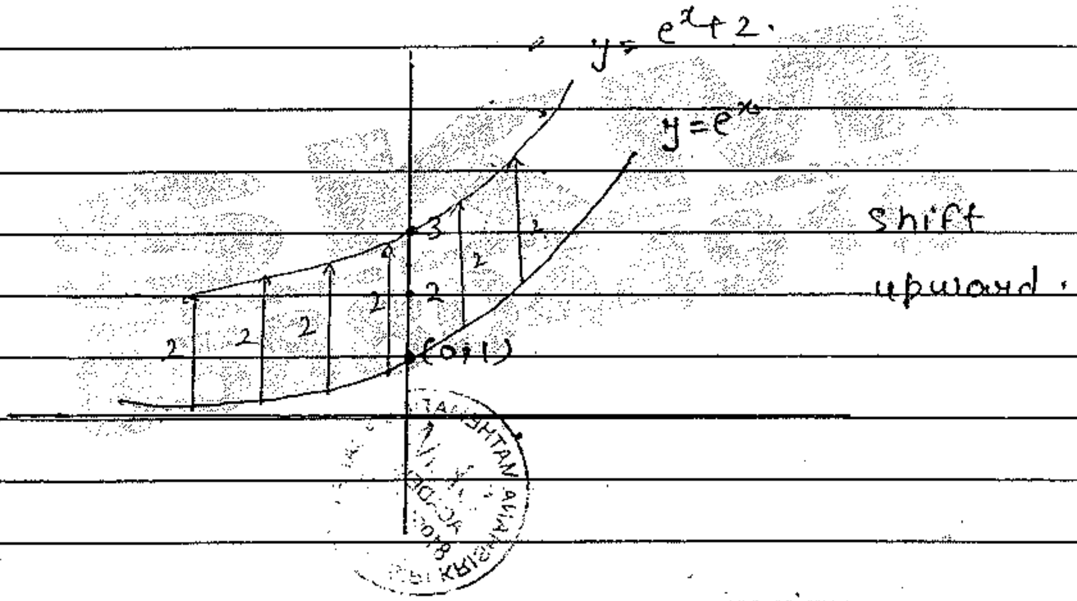
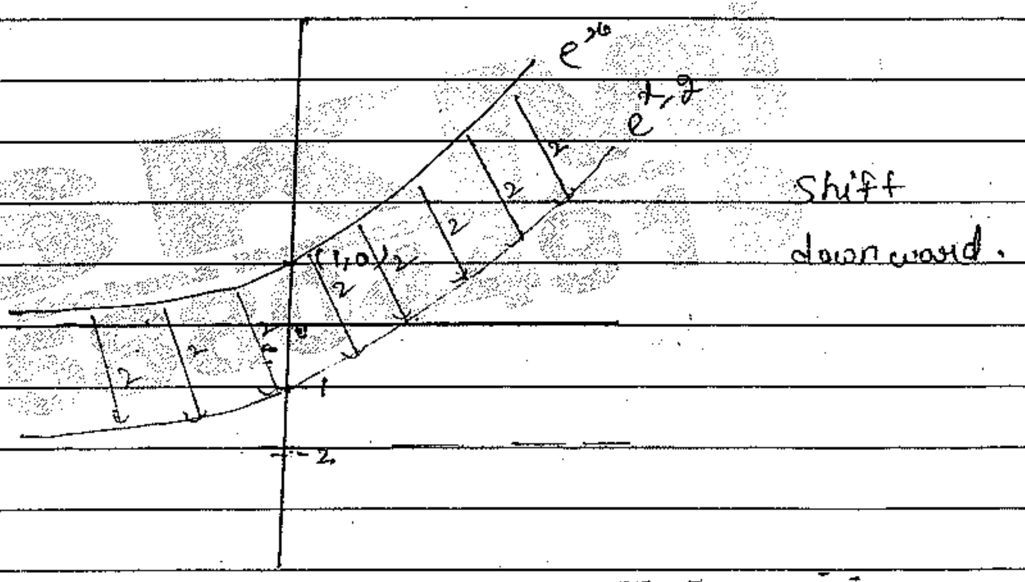
$f(x)$ transfoorm to $f(x)+a$ if $a>0$ (Shift upwoord)
$f(x)$ transform to $f(x)-a$ if $a>0$ (Shift downward

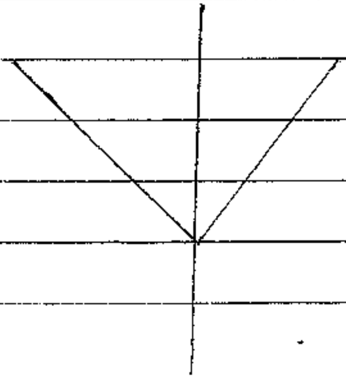
Ex:- $f(x) = e^{x}+2$.


$y = e^{x}+2$.
$y = e^{x}$
3
2
2
2
2
2
2
2
(0,1)
Shift
upward.

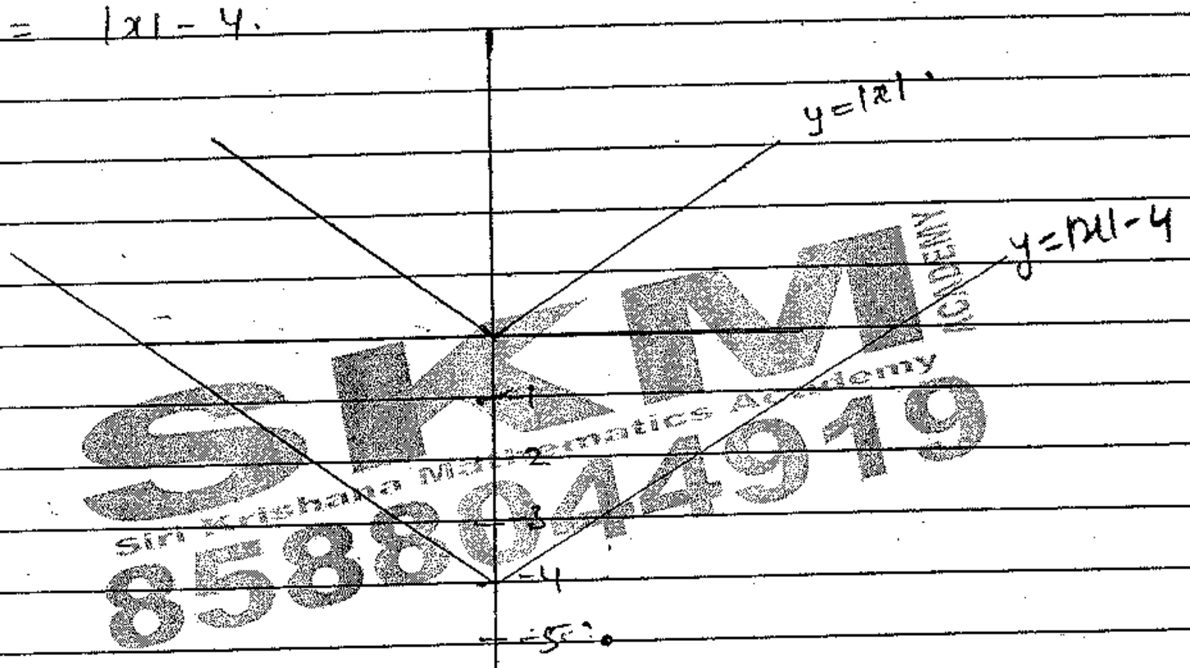
Ex:- $f(x) = e^{x}-2$


e^{x}
$e^{x}-2$
Shift
downward.
(1,0)
2
2
2
2
2
2
2
1
-2

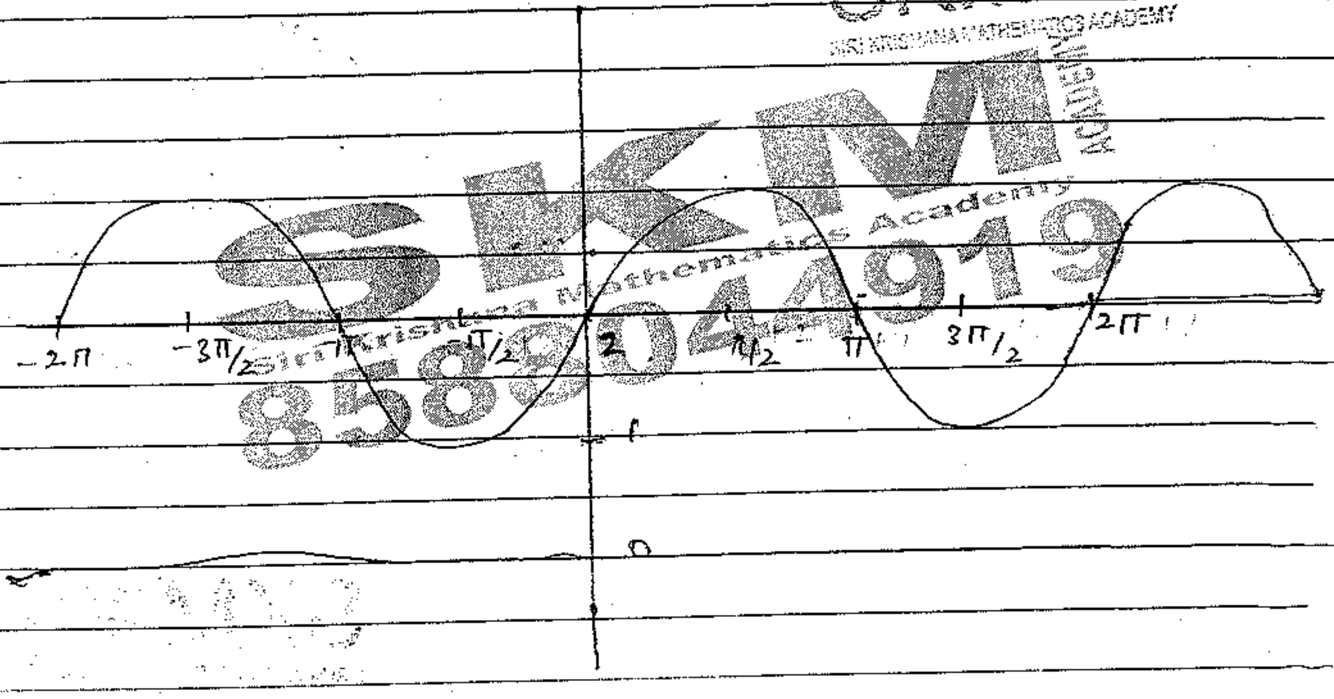
Ex:- $f(x) = |x|$.



Ex:- $f(x) = |x| - 4$.



Ex:- $f(x) = \sin x + 2$.



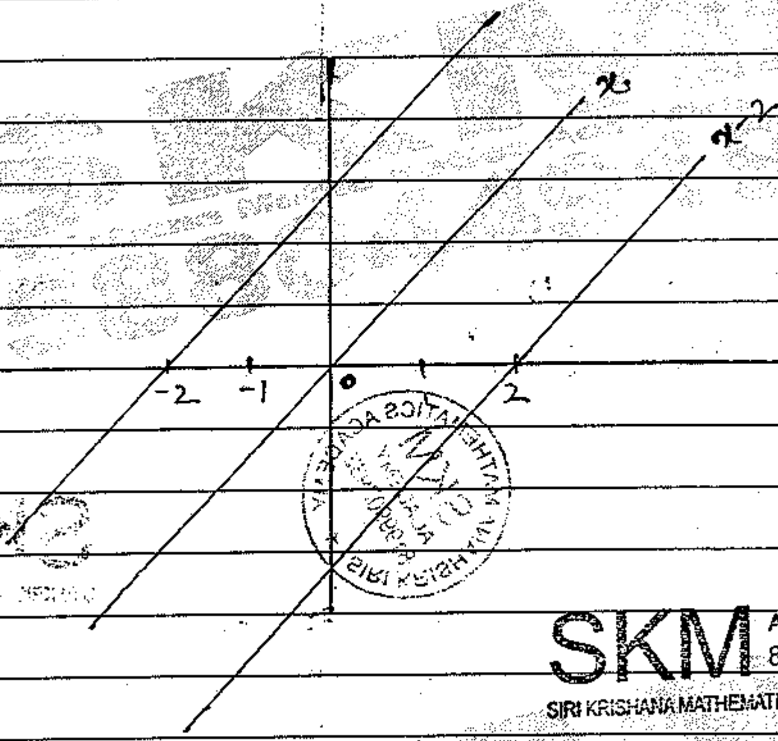
# $f(x)$ transform to $f(x+a)$ if $a>0$ shift to left.

f(x) transform to $f(x-a)$ if $a>0$ shift to Right.

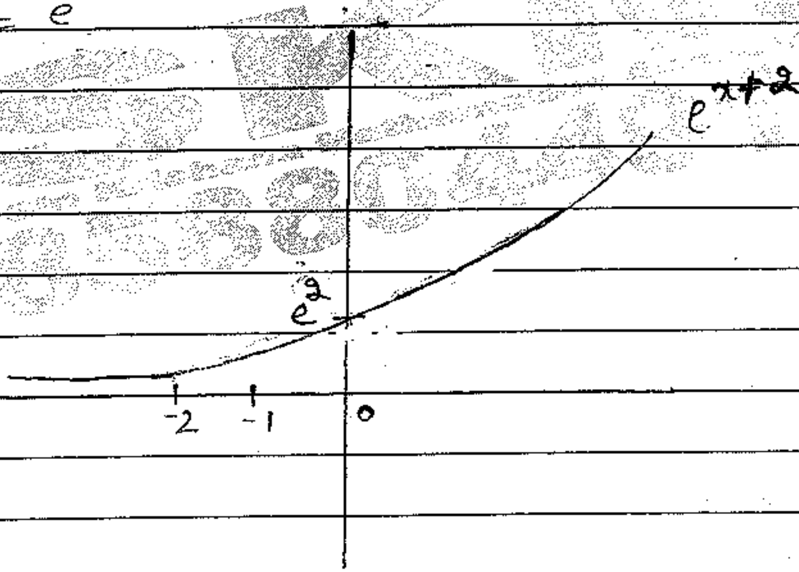
Eg:- $f(x)= x$

$f(x)= x+2$

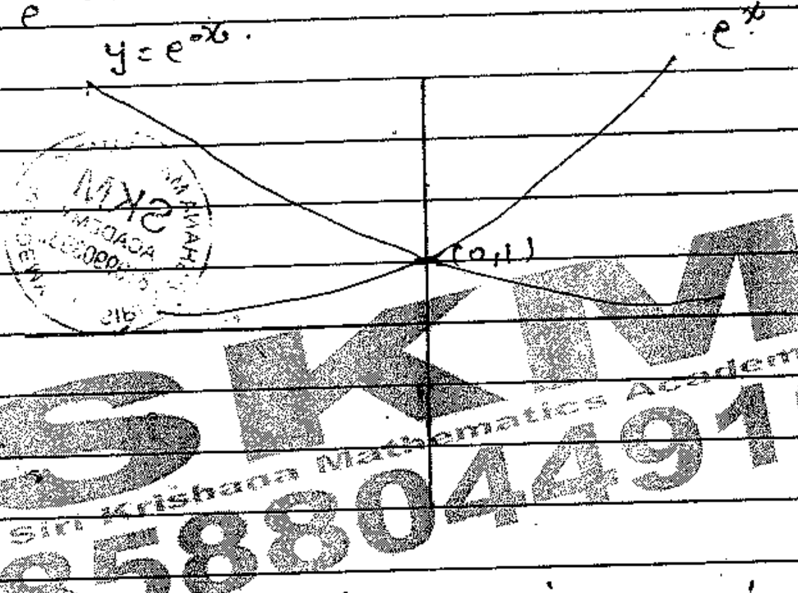
$f(x)= x-2$.


x+2
x
x-2
-2
-1
0
1
2

Eg:- $f(x) = e^{x+2}$

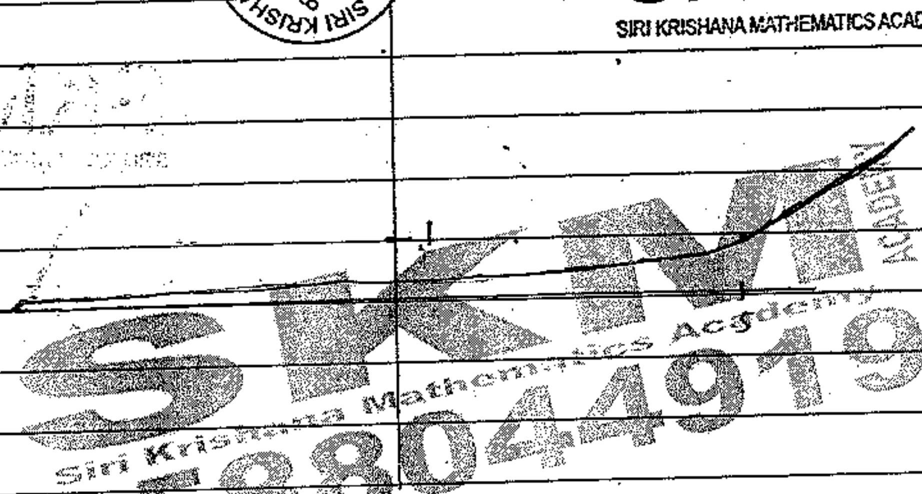


$f(x) = e^{-x}$.



$y = e^{-x}$ is the image in y-axis as plane mirror for $y = e^{x}$

Eg:- $f(x) = e^{x-5}$

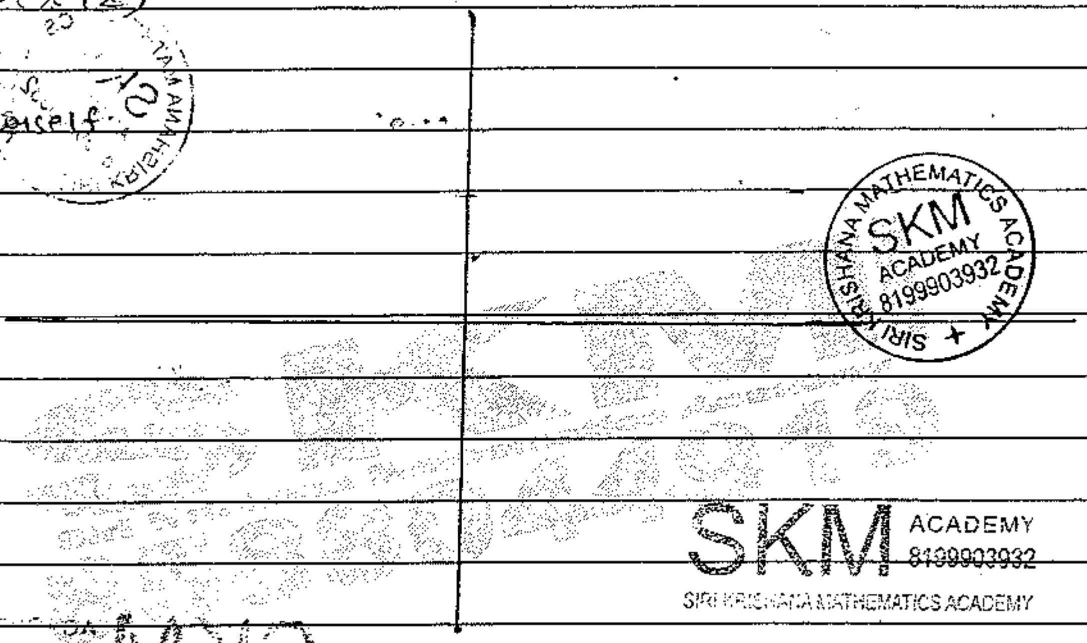


Eg:- $f(x) = \sin(x-2)$

Try yourself.

$f(x) = \cos(x+2)$

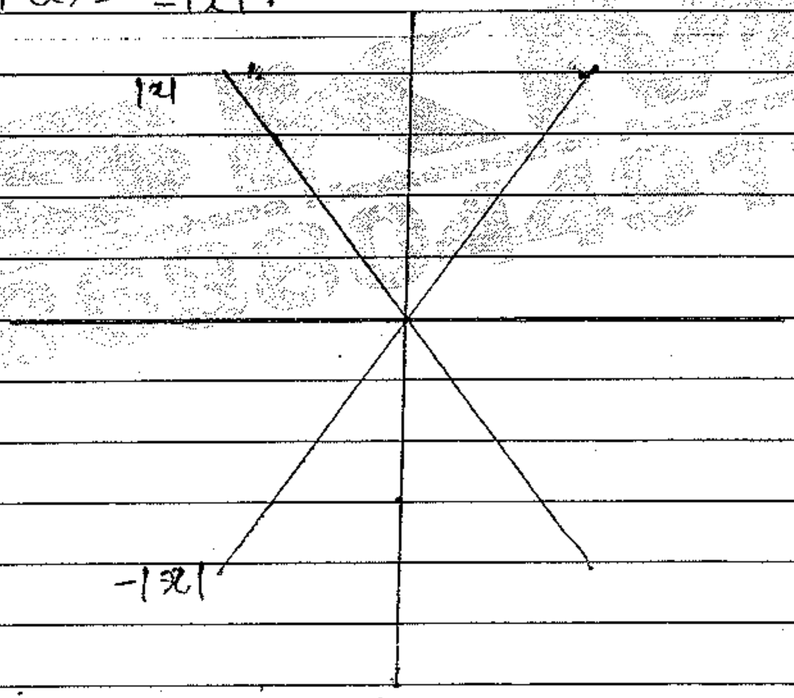
Try Yourself



f(x) transformation −f(x) is the mirror image of x-axis.

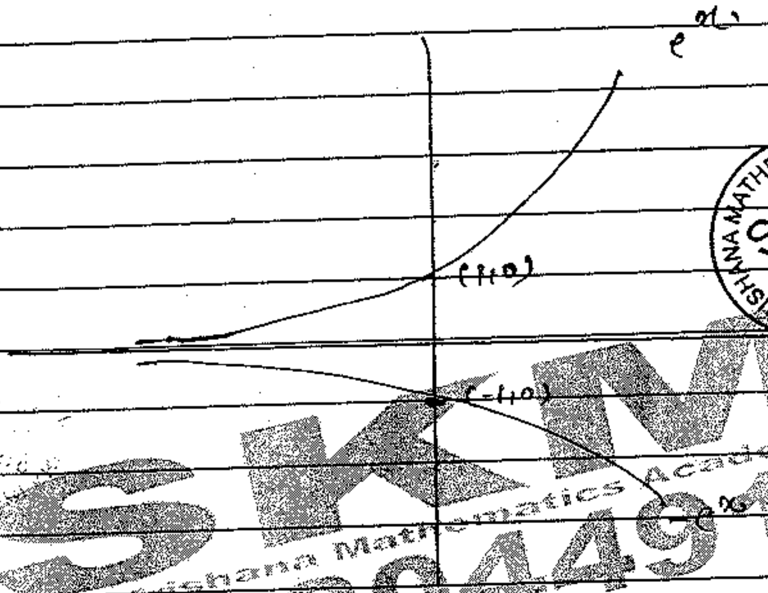
$|x| \longrightarrow -|x|$.

Ex:- $F(x) = -|x|$.

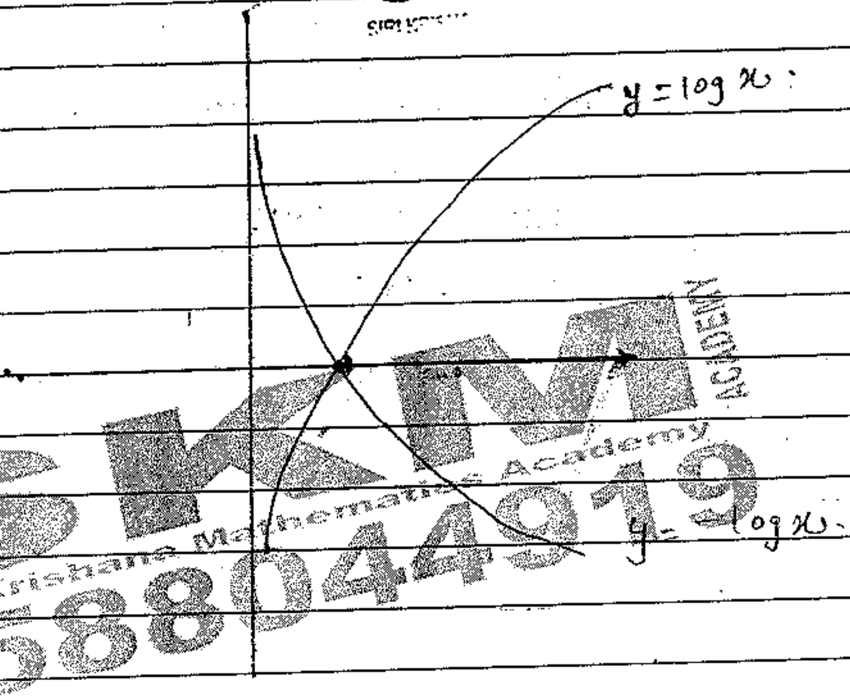


$f(x) = e^{x}$

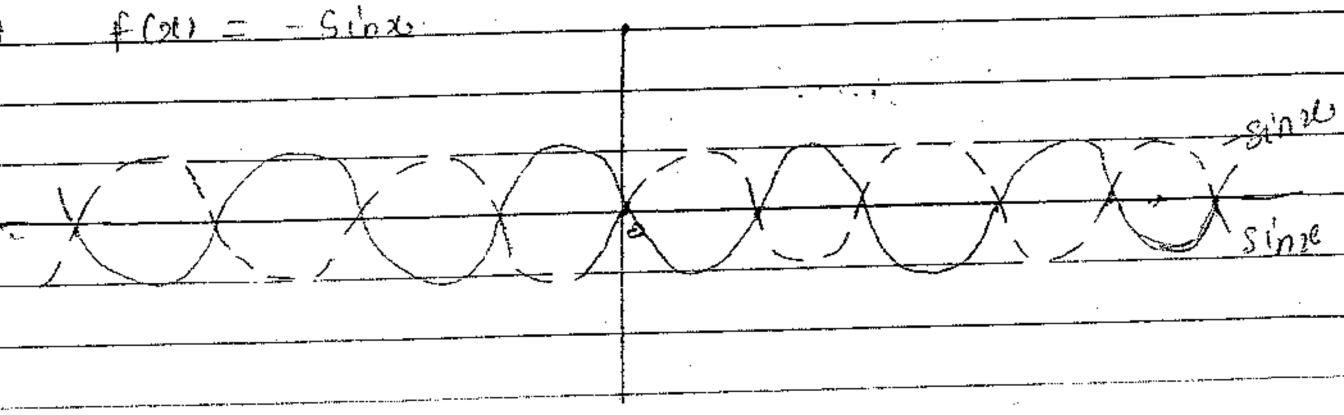
$f(x) = -e^{x}$



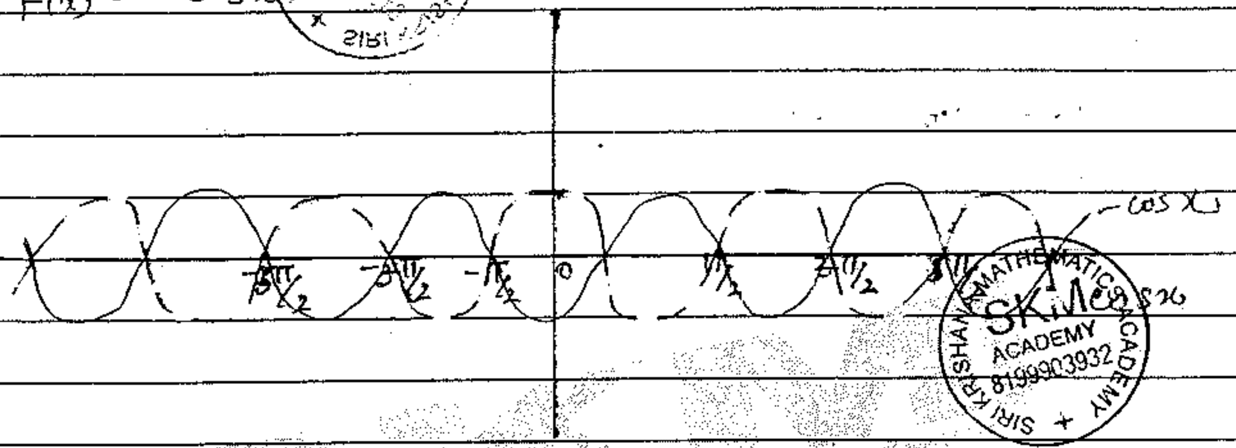
$f(x) = -\log x$



$f(x) = -\sin x$

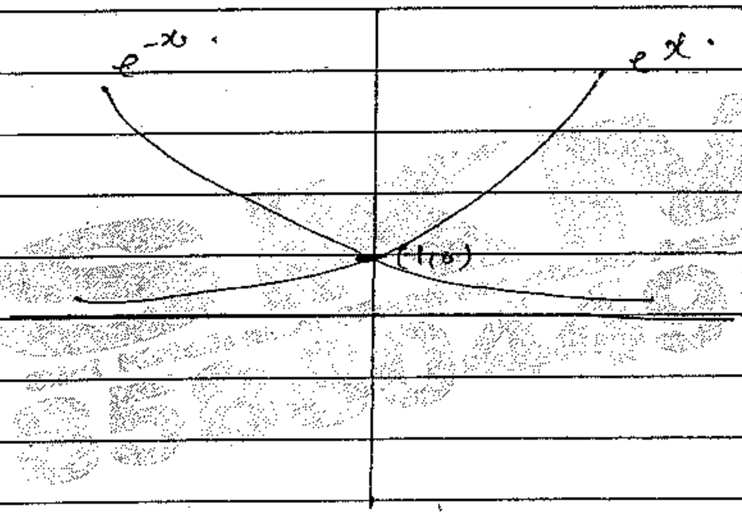


$f(x) = -\cos x$

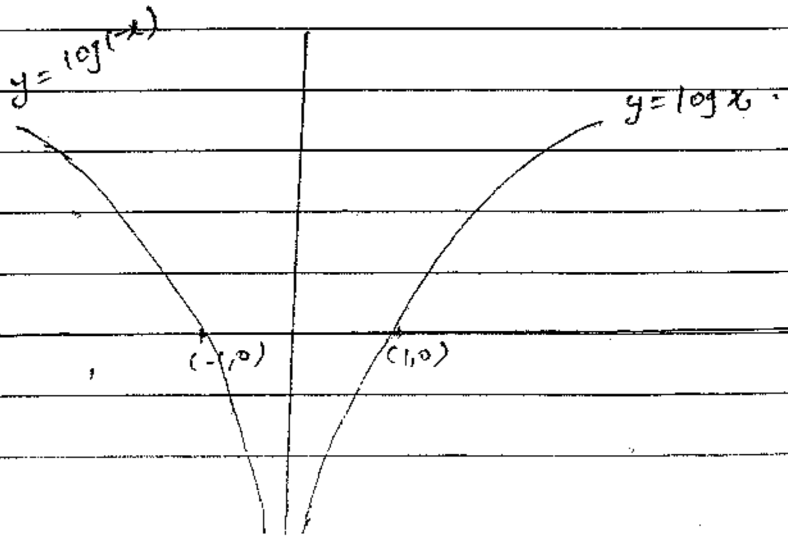


Transformation $f(x)$ into $F(-x)$, Mirror image of Y-axis.

$f(x) = e^{x}$
$f(-x) = e^{-x}$.



$f(x) = \log x$
$f(-x) = \log(-x)$



$f(x) = \sin(-x)$.

Try yourself.

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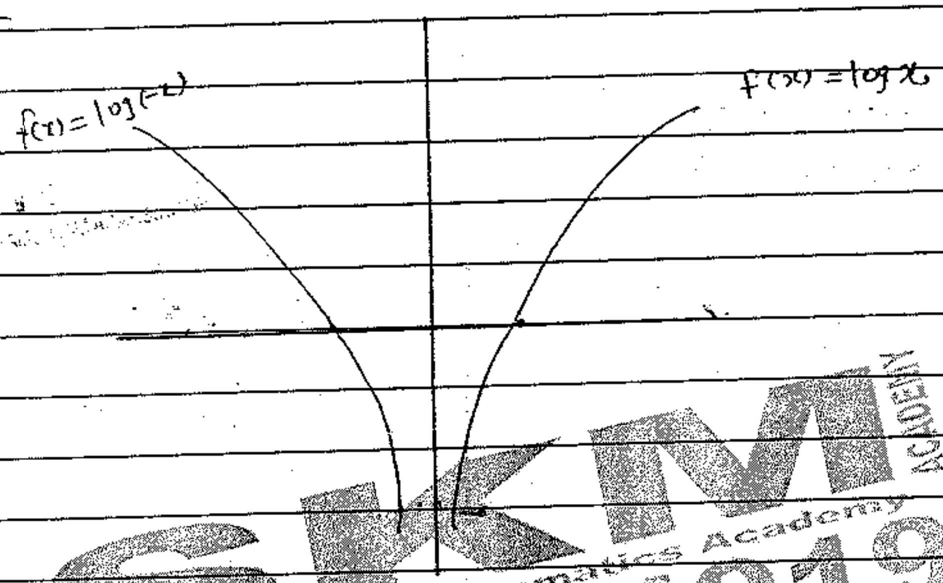
$\log|x|$ =



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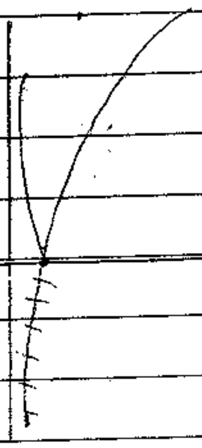
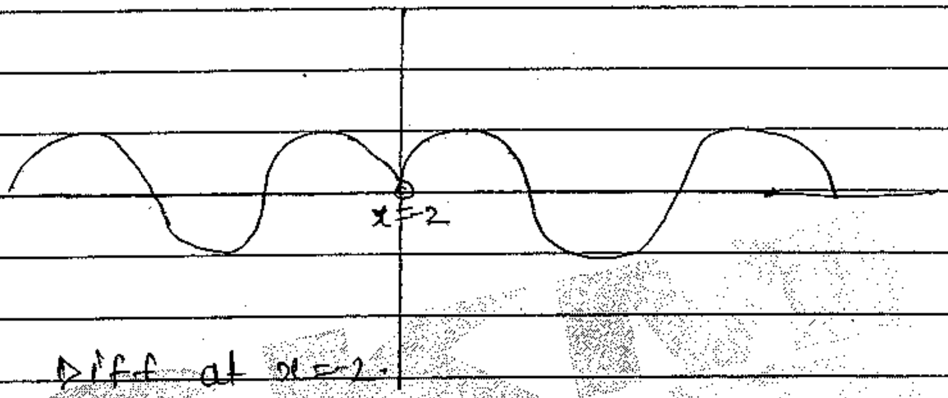
$f(x) = |\log x|$



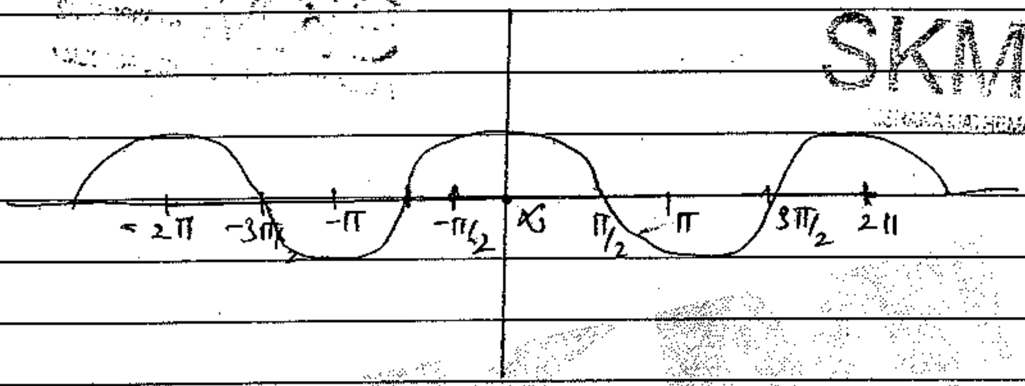
$f(x) = \sin|x+2|$



Not Diff at $x=-2$.

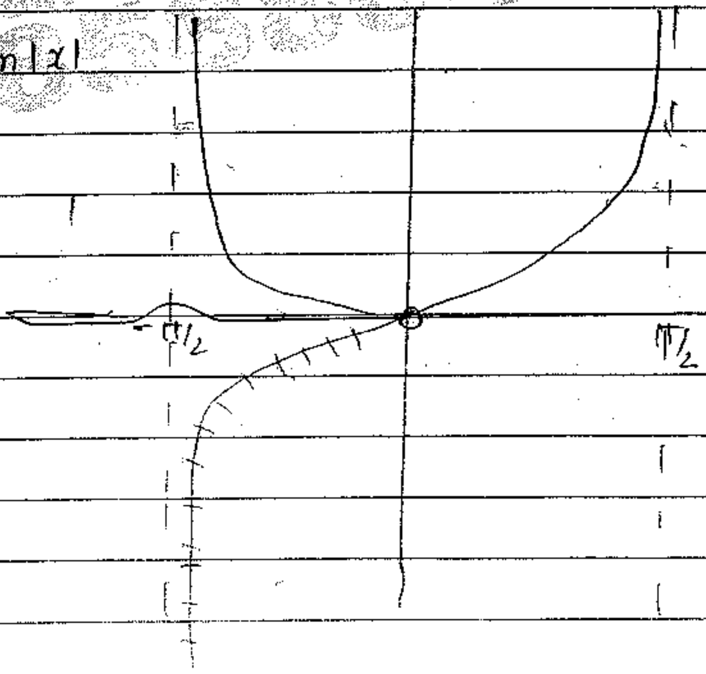
$f(x) = \cos|x+2|$ $\alpha=-2$

$\cos|x-2|$ $\alpha=2$.



$|\cos x|$ is Not diff at $x = (2n+1)\pi/2$.

$f(x) = \tan|x|$



Diff everywhere excep at $x=0$.

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Differentibality.

The Derivative and Mean Value Theorems :-

Derivative of a function at a point.

Let $f: I \longrightarrow \mathbb{R}$ be a function and $c \in I$, then 'f' is said to be derivable or differentiable at 'c' if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ exist.

Let $\phi(x) = \frac{f(x)-f(c)}{x-c}$ then $\lim_{x \to c} \phi(x)$ is denoted by $f'(c)$.

NOTE ⇒ c is limit point of I and belong to I

Note :- if $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0} \frac{f(c-h)-f(c)}{-h}$ exist then limit is called left hand derivative of f at 'c' and denoted by $f'(c-0)$ or $f'(-c)$ or $Lf'(c)$.

Note :- if $\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ exist then limit is called Right hand derivative of 'f' at 'c' and denoted by $f'(c+0)$ or $f'(+c)$ or $Rf'(c)$

Note :- The derivative $f'(c)$ exists iff $Lf'(c) = Rf'(c)$

Note :- 'f' is said to be derivable [illegible]b) if $f'(c)$ exists ∀ c s.t $a<c<$ [illegible]

Note :- 'f' is said to be derivable in $[a,b]$ if

i) $f'(c)$ exist $\forall$ $c \in (a,b)$

ii) $Rf'(a)$ exist.

iii) $Lf'(b)$ exist.

Note :- Let 'f' be a function whose Domain is an interval I. Let S be the set of all those points x of I at which $f'(x)$ exist. clearly $S \subseteq I$. If $S \neq \phi$ then the function 'f' with Domain S is called the derivative of f and $f', f'', f''', ----- f^{(n)}$ are first, second, third ... nth derivative of 'f'.

Note :- $f'(c) = Df(c) = \frac{df}{dx}(c) = \left.\frac{dy}{dx}\right|_{x=c}$

where $y = f(x)$ [Similar Notation for $f'(c)$].

Examples :-

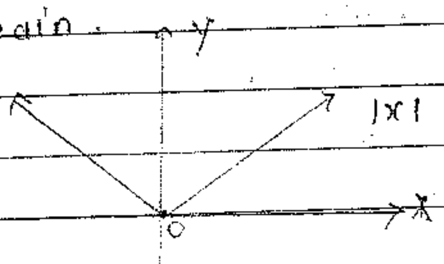
① $f(x) = |x|$; $\forall$ $x \in \mathbb{R}$.

$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = 1 = R.H.D.$

$\lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|h| - 0}{-h} = -1 = L.H.D$

So 'f' is Not diff. at $x = 0$ b/coz $Rf'(0) \neq Lf'(0)$

. So function is Not Differentiable only $x = 0$ in their Domain.



There is a peak in the graph of $|x|$ at $x=0$. So function is Not diff. at $x=0$.

② $f(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x} & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$

is differentiable at $x=0$.

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x} = \lim_{x\to 0} x \cdot \sin\frac{1}{x} = 0$$

Limit exist at $x=0$ so $f(x)$ is differentiable at $x=0$.

③ $f(x) = \begin{cases} x \cdot \sin\frac{1}{x} & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$ is Not diff. at $x=0$

④ $f(x) = \begin{cases} x^{1+t} \sin\frac{1}{x} & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$

Now $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^{1+t} \sin\frac{1}{x}}{x}$

$$= \lim_{x\to 0} x^t \cdot \sin\frac{1}{x}$$

if $t \leq 0$ then $\lim_{x\to 0} x^t \sin\frac{1}{x}$ doesn't exist and.

if $t > 0$ then $\lim_{x\to 0} x^t \cdot \sin\frac{1}{x}$ exists. is differentiable at $x=0$ if $t > 0$

5. $$f(x) = \begin{cases} x \cdot \tan^{-1} 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$$\lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{h \cdot \tan^{-1} 1/h}{-h} = -\tan^{-1}(\infty) = -\frac{\pi}{2}$$

L.H.D

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \cdot \tan^{-1} 1/h}{h} = \tan^{-1}(\infty) = \frac{\pi}{2} = R.H.D$$

So Here R.H.D ≠ L.H.D So $f(x)$ is Not diff at $x=0$

Note :- If 'f' is Linear function or a constant function over the various Subintervals.
then 'f' is continuous and differentiable over each Subinterval and the only doubtful points are the breaking points.

Eg:- $f(x) = |x-1| + |x-2|$ in $[0, 3]$

$$f(x) = \begin{cases} 1-x+2-x = 3-2x, & \text{if } 0 \leq x \leq 1 \\ x-1+2-x = 1, & \text{if } 1 \leq x \leq 2 \\ x-1+x-2 = 2x-3, & \text{if } 2 \leq x \leq 3. \end{cases}$$

So $f(x)$ is going to be break at $x = 1$ & 2 so there are only two points where we Need to Check continuity and differentiability of $f(x)$

So $f(x)$ is cts at $x = 1$ & 2 but Not diff at $x = 1$ & 2.

Note :- Derivative of 'f' at 'c' is the slope of tangent to the curve $y = f(x)$ at a point $(c, f(c))$.

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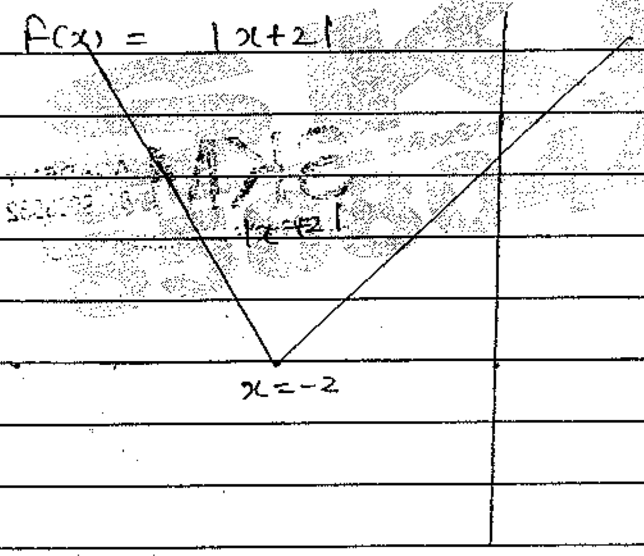
Note:- if 'f' is differentiable on (a,b), then at any point of (a,b); curve has a unique tangent.

Note:- If there is a point in Domain at which curve have more than one tangent then 'f' is Not differentiable there at.

Eg:- $f(x) = |x+2|$



So ~~At~~ $x=-2$ there is a peak in curve. ~~we~~ So at $(2, f(-2))$ point we can draw more than one tangent So $f(x)$ is Not diff at $x=-2$.

Eg:- $f(x) = \sin x$ then $|\sin x|$ will Not be diff. at $x = n\pi$, $n \in Z$.

Ex:- $f(x) = |\sin x| + |\cos x|$.

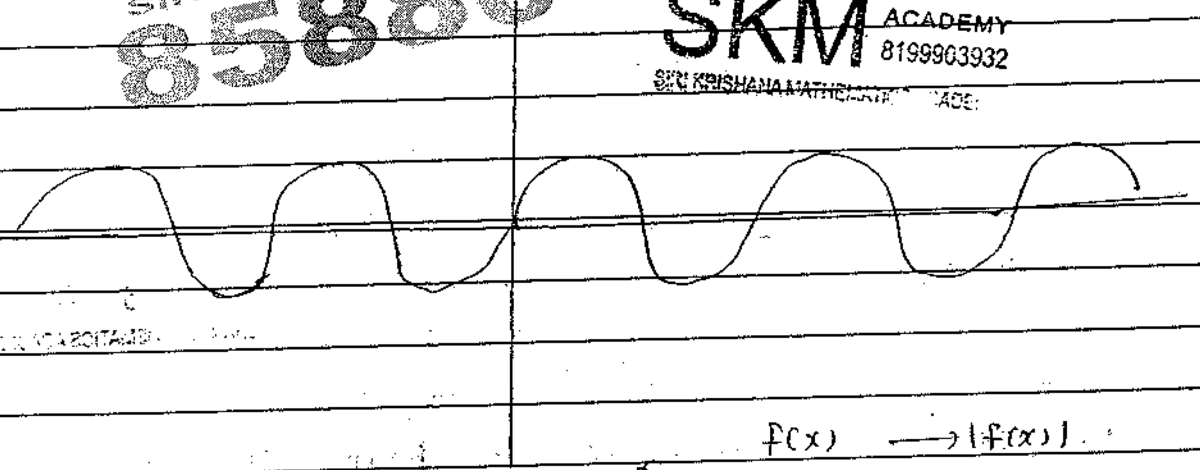
$f(x)$ is Not differentiable at the point where $\sin x$ and $\cos x$ is zero.

$S_1 = \{n\pi,\ n \in Z\}$ and $S_2 = \{(2n+1)\frac{\pi}{2},\ n \in Z\}$

Here S_1 is the set of points where $\sin x$ is zero.
and S_2 is the set of points where $\cos x$ is zero.

So $f(x)$ is Not differentiable at $x = a$ where
$a \in S_1 \cup S_2$ i.e. $a \in \{n\pi, n \in Z\} \cup \{(2n+1)\frac{\pi}{2}, n \in Z\}$

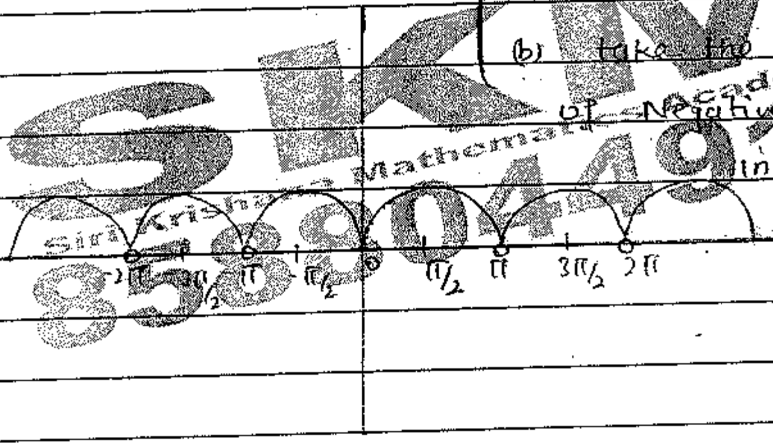
$f(x) = \sin x$.



$f(x) \longrightarrow |f(x)|$.

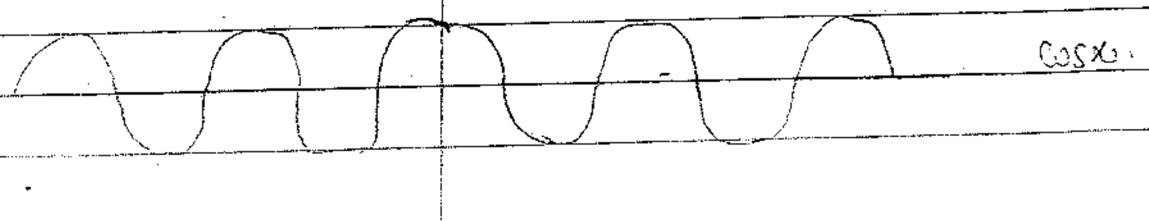
(a) leave the +ve Part of $f(x)$ as it is
(b) take the mirror image of Negative Part of $f(x)$ in the x-axis as plane mirror.

$f(x) = |\sin x|$

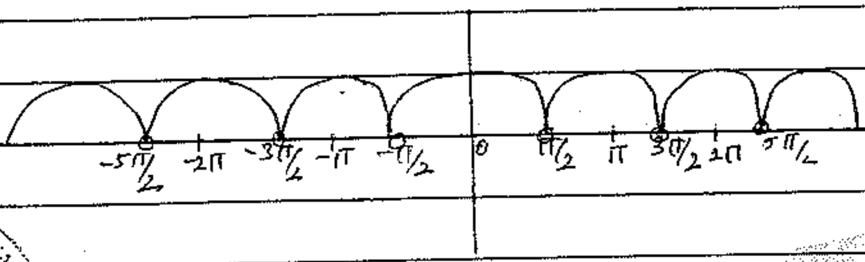


function is Not diff $x = n\pi$.

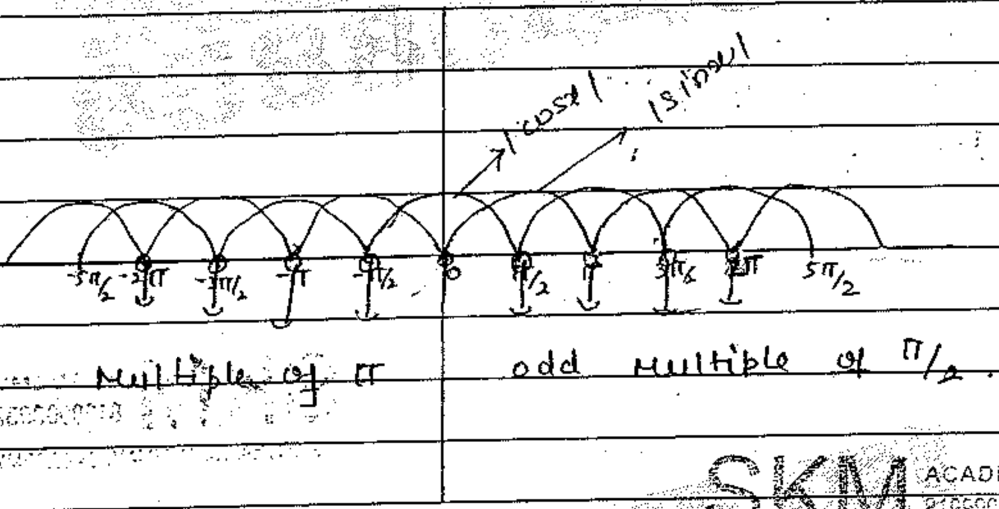
$f(x) = \cos x$.



$f(x) = |\cos x|$


$-5\pi/2$
-2π
$-3\pi/2$
$-\pi$
$-\pi/2$
0
$\pi/2$
π
$3\pi/2$
2π
$5\pi/2$

$f(x) = |\sin x| + |\cos x|$


$|\cos x|$
$|\sin x|$
$-5\pi/2$
-2π
$-3\pi/2$
$-\pi$
$-\pi/2$
0
$\pi/2$
π
$3\pi/2$
2π
$5\pi/2$

Multiple of π | odd Multiple of $\pi/2$.

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Example $f(x) = x \cdot |x| \;\forall\; x \in \mathbb{R}$ then.

$$f(x) = \begin{cases} x \cdot (x) = x^2 \;; & x > 0 \\ x(-x) = -x^2 \;; & x \leq 0 \end{cases}$$

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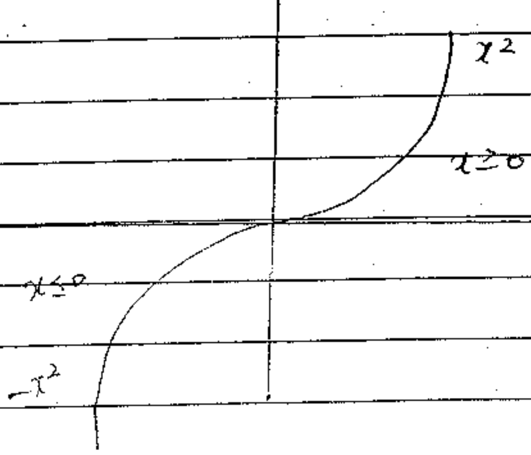
x^2
$x \geq 0$
$x \leq 0$
$-x^2$

So $f(x)$ is differentiable at $x=0$.

Note:- $\sin x$, $\cos x$, e^x, $P(x)$ [Poly. function] all are differentiable at every point of it's dod. where dod $= \mathbb{R}$.

Note $\sin|x|$, is Not diff only at $x=0$

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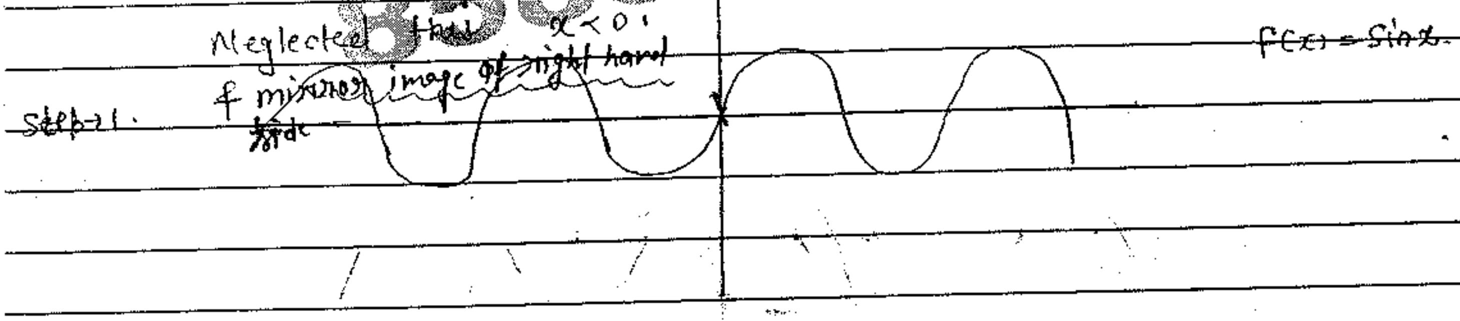

for $f(x) = |\sin x|$.

$f(x) \longrightarrow f(|x|)$.

(i) Leave the graph Right side of the y-axis as it is

(ii) Take the image of $f(x)$ in the Right of y-axis as the plane mirror and the graph of $f(x)$ lying leftward of the y-axis (if it exist) is omitted.

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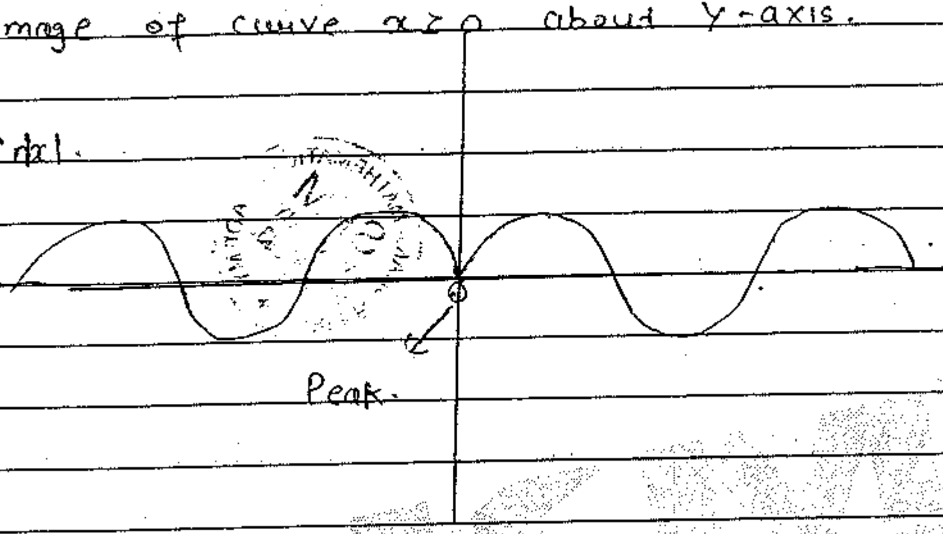
$\boxed{\text{OR}}$

Neglected the curve for $x<0$ and take the images of curve for $x>0$ about y-axis.

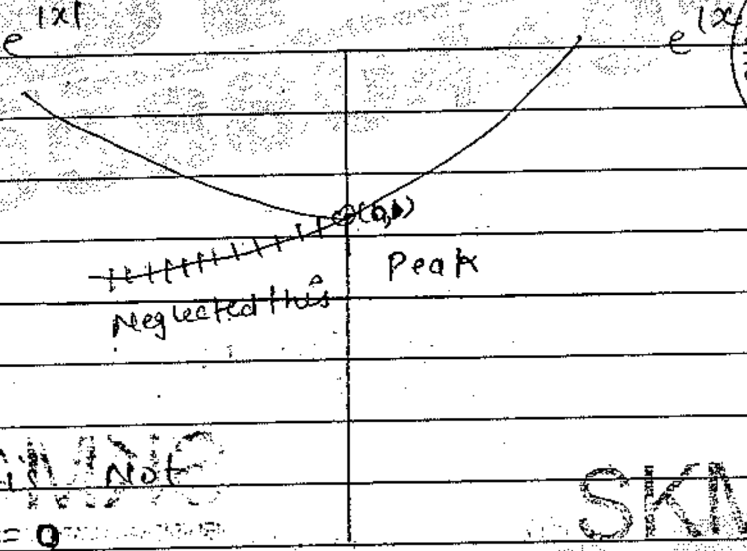
The image of curve $x \geq 0$ about Y-axis.

Step→2

$f(x) = \sin|x|$.

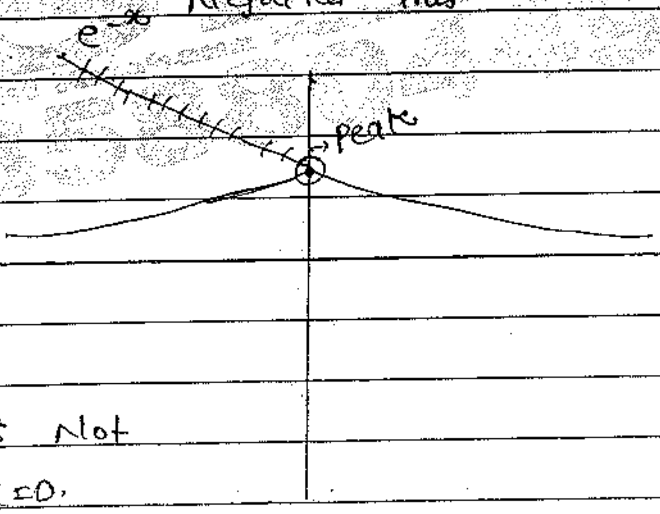


Similarily $e^{|x|}$


$e^{|x|}$
(0,1)
Peak
Neglected this

So $e^{|x|}$ is Not
Diff at $x = 0$

Similarly $e^{-|x|}$ Neglected this



So $e^{-|x|}$ is Not
diff at $x=0$.

Q) Find a function from ℝ to ℝ which is diff. at exactly one point.

$$f(x) = \begin{cases} x^2 , & x \in \mathbb{Q} \\ 0 , & x \in \mathbb{Q}^c \end{cases}$$

when x is Rational then $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x = 0$

when x is irrational then $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} 0 = 0$

Hence $f(x)$ is diff at $x=0$.

Now let us assume that 'f' is diff at $x=a\neq 0$. then,

if $x=a$ is Rational $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^2-a^2}{x-a} = 2a$.

if x is irrational $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 0}$ [illegible] $\frac{0}{x-a}$ Not defined

Hence 'f' is Not defined at any other point $a\neq 0$. of ℝ. Hence '0' is the only point of differentiability.

Q) $f(x) = \begin{cases} x^2+3x+a & \text{if } x\le 1 \\ bx+2 & \text{if } x>1 \end{cases}$ is given to be

differentiable for every x, find a & b.

Soln:- $f(x)$ is diff $\Rightarrow$ $f(x)$ is cts [at $x=1$]

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^2+3x+a)$

$= \lim_{h\to0} ((1-h)^2+3(1-h)+a) = 4+a$

& $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} bx+2 = \lim_{h\to0} b(1+h)+2 = b+2$

Also $f(1) = 1+3+a = 4+a$.

∵ f is cts at $x=1$ so

$4+a = b+2$

$\Rightarrow a-b+2=0$ —(i)

Now $Lf'(1) = \lim_{h\to0} \frac{(1-h)^2+3(1-h)-4}{-h} = \lim_{h\to0} \frac{h^2-5h}{-h}$

$Lf'(1) = 5$

$Rf'(1) = b$

f is differentiable at $x=1$ so

$Rf'(1) = Lf'(1)$

$\Rightarrow b = 5$

From (i) $a = 3$

Hence $\boxed{a=3 \ \& \ b=5}$

Result :- if 'f' and 'g' both are differentiable at $x=a$ then $f+g$, $f-g$, $f\cdot g$ are differentiable. and Have derivatives & given by

i) $(f+g)'(a) = f'(a)+g'(a)$

ii) $(f\cdot g)'(a) = f'(a)\cdot g(a) + g'(a)\,f(a)$

iii) $(f-g)'(a) = f'(a) - g'(a)$

iv) If $g'(a) \neq 0$ then f/g is differentiable at 'a' and has derivative given by

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)\, f'(a) - f(a)\, g'(a)}{[g(a)]^2}$$

Result:- if $f(x)$ is diff at $x=a$ and $g(x)$ is diff at $f(a)$ then $h(x) = (g \circ f)(x)$ is differentiable at 'a' and have the derivative

$$h'(a) = g'[f(a)] \cdot f'(a)$$

Note:- If $f(x) + g(x)$ or $f(x) \cdot g(x)$ be derivable at 'a' then $f(x)$ and $g(x)$ are Not Necessarily derivable at 'a'.

Eg:- $f(x) = \begin{cases} 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$, $g(x) = \begin{cases} -1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Here $f+g$ is diff at 0 but $f(x)$ & $g(x)$ are Not.

Eg:- $f(x) = \begin{cases} x \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ & $g(x) = \begin{cases} x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$f(x) \cdot g(x)$ is diff at 'o' but $f(x)$ is Not.

Thm:- If a function is derivable at 'c' then it is continuous at $x = $'c'.

Note:- Differentiability at a point $\Rightarrow$ continuity at that point. and
$f(x)$ is cts at a point $\not\Rightarrow$ Derivability at that point

Eg:- $f(x) = |x|$ is cts at '0' but Not differentiable.

Properties of Differentiability :-

1> if we have curve $y = f(x)$ then differentiability at $x = a$ implies the existence and uniqueness of the tangent to the curve $y = f(x)$ at $x = a$. And then slope of this tangent is given by $f'(a)$

2> if $f'(a) > 0$ then we say 'f' is in increasing mode at $x = a$.

3> if $f'(a) < 0$ then we say 'f' is in decreasing mode at $x = a$.

4> if $f'(a) = 0$, then we say f is in stationary mode at $x = a$ i.e at $x = a$, curve $y = f(x)$ has a tangent parallel to x-axis. And this point $x = a$ is called stationary point of f.

ie If $f'(a) = 0$ then $\exists$ a some Nbd of a. S.t $f'(x) > 0$ for some values x of Nbd of a. and $f'(x) < 0$ for some values x of Nbd of a.

i.e 'f' is Not Monotonic function.

5> If $f(x)$ is differentiable on I and $f'(x) > 0$ $\forall x \in I$, then f is Strictly increasing.

∴ Hence one-one.

⑥ if f(x) is differentiable on I and $f'(x) < 0$ $\forall x \in I$, then f is strictly decreasing. Hence one-one.

Ex:- Check $f(x) = \frac{1}{2}x^2 + \log x + \cos x$ is increasing or decreasing function in domain $(0, \infty)$

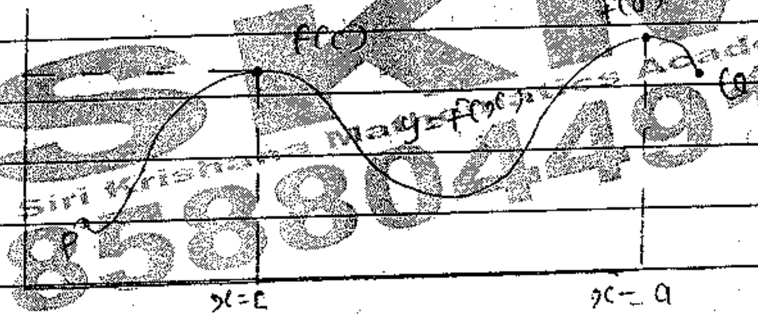
$\Rightarrow f'(x) = x + \frac{1}{x} - \sin x$

$f'(x) > 0 \; \forall \; x \in (0, \infty)$.

b'coz $x + \frac{1}{x} > 0$ and $\sin x \leq 1$

So f(x) is increasing in $(0, \infty)$.

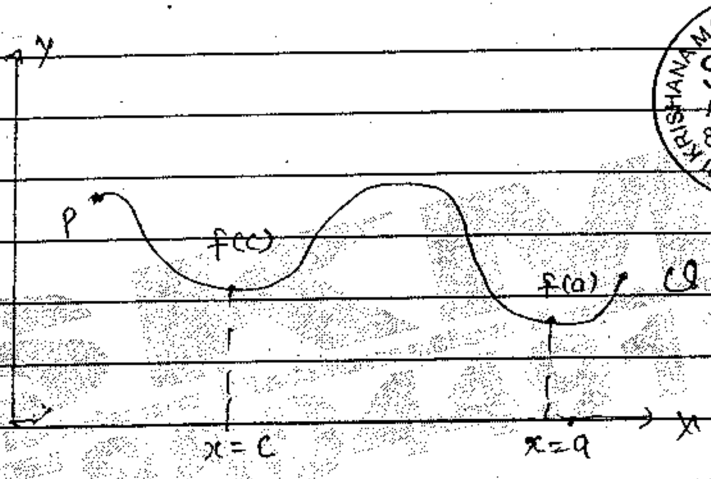
7) Local Maxima:- Let 'f' be defined on I and $c \in I$. then 'f' has local Maxima at $x = c$ if $\exists$ a $\delta > 0$ s.t $f(x) \leq f(c) \; \forall \; x \in (c - \delta, c + \delta)$



fig(A)

by graph easily shown that $f(c) \geq f(x) \; \forall$ value of x in Nbd of 'c', Hence 'c' is the point of local Maxima.

⑧ Local Minima :- f is defined on I and $c \in I$ then f has local Minima at $x=c$ if $\exists\ \delta > 0$ s.t $f(x) \geq f(c)\ \forall\ x \in (c-\delta, c+\delta)$.



fig(B)

Note: In fig(A), at $x=a$ is the point of Absolute Maximum i.e $f(a) \geq f(x)\ \forall\ x \in$ Domain where PQ is the graph of $f(x)$.

Note!:- In fig(B), $x=a$ is the point of Absolute minima. i.e $f(a) \leq f(x)\ \forall\ x \in$ Domain, where PQ is graph of $f(x)$.

Results about Extrema [Maxima and Minima]

i) if 'f' is Diff at $x=c$ and 'c' is the point of extrema (minima or Maxima) of $y=f(x)$ then $f'(c)=0$ converse Not true.

Eg:- $f(x)=x^3$, $f'(0)=0$, but 'f' is strictly increasing and hence it has No local extremum at $x=0$.

Note:- Let 'f' be a function (Need not diff at 'a') and f(a) is local extremum then $f'(a)=0$ or else $f'(a)$ does not exist where $a \in I$.

② If $f'(a) > 0$ where $a \in I$ and 'f' is diff on I then $f'(x)$ is Retain the sign of $f'(a)$ in some Nbd of 'a'.

③ If 'f' is differentiable on I and $\exists\ a, b \in I$ s.t $f'(a) \cdot f'(b) < 0$ then $\exists\ c \in (a,b)$ s.t $f'(c) = 0$

Darboux's Theorem:- Let 'f' is differentiable on I and $a, b \in I$ If $k \in R$ s.t $f'(a) < k < f'(b)$ then $\exists$ 'c' $\in (a,b)$ s.t $f'(c) = k$.

(or)

If f is has derivative at every point of $[a,b]$ then 'f'' takes every value b/w $f'(a)$ and $f'(b)$.

i.e The image of interval under $f'(x)$ either interval or singleton.

Primitives:- A derivable function F, if exist s.t its derivative F' is equal to the given function f, is called Perimitive of f.

i.e F is differentiable $\forall\ x \in I$ & $F'(x) = f(x)$ [say]. then F is called Permitive of 'f' on I.

Note:- Sufficient condition for a function to admit of a Primitive is that it is continuous.

Thus Every continuous function 'f' Possesses a Primitive F, where $F(x) = \int_a^x f(t)\,dt$.

Note:- Continuity is Not Necessary condition for a function to have a Primitive.

i.e. function possessing primitives are Not Necessarily continuous.

Eg:- $f : [0,1] \longrightarrow \mathbb{R}$ s.t

$$f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

It has one Primitive F, where

$$F(x) = \begin{cases} x^2 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

Clearly $F'(x) = f(x)$ but $f(x)$ is Not Differentiable at $x=0$ i.e $f(x)$ is Not cts on $[0,1]$.

Result:- (i) 'f' is differentiable on I and $\alpha \in I$ s.t $f'(\alpha)=0$, then α is point of local Maxima if $f'(\alpha-h)>0$ and $f'(\alpha+h)<0$, and

(ii) 'f' is differentiable on I and $\alpha \in I$ s.t. $f'(\alpha)=0$, then α is point of local Minima if $f'(\alpha-h)<0$ and $f'(\alpha+h)>0$.

Convex function:- Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be convex on I if for any t satisfying $0 \leq t \leq 1$ and any points x_1, x_2 in I, we have

$$f[(1-t)x_1 + t x_2] \leq (1-t) f(x_1) + t f(x_2)$$

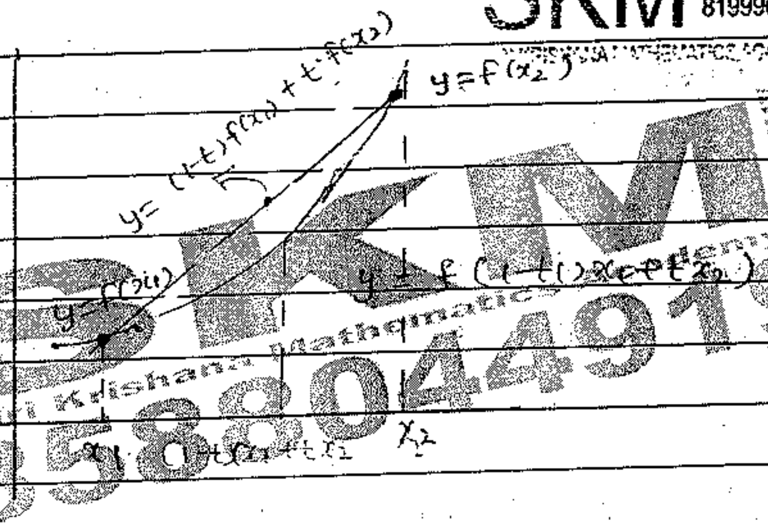
(or)

$$F[\lambda_1 x_1 + \lambda_2 x_2] \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) \text{ where}$$

$$\lambda_1 + \lambda_2 = 1 \ \& \ \lambda_1, \lambda_2 \geq 0$$

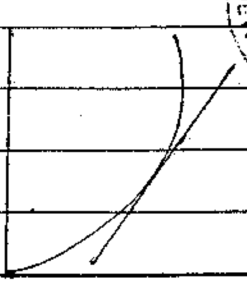
(or)

if 'f' is convex on I and if $x_1, x_2 \in I$ then the chord joining any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of 'f' lies above the graph of f.

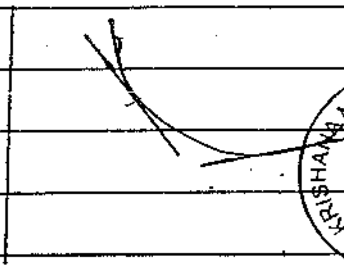


(or)

'f' is said to be convex on I if any point of interval I, the tangent lies below the curve $y = f(x)$.



(i)



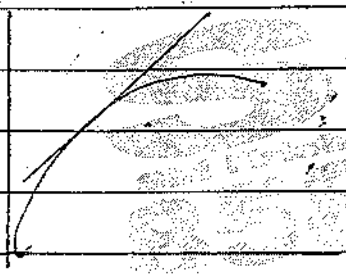
(ii)

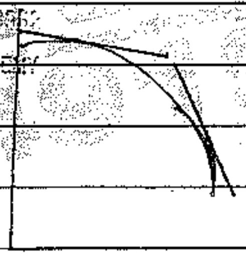
Both are convex function's graph.

Concave function :- if 'f' is concave on I and if $x_1, x_2 \in I$ then the Chord joining any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of 'f' lies below the graph of $y = f(x)$.

(or)

'f' is said to be concave on I if at any point of interval I, the tangent above the curve.





All graphs are the concave function.

Result: If 'f' is twice differentiable function on an interval I then 'f' is convex function if and only if $f''(x) > 0 \ \forall x \in I$ or $f''(x) \geq 0$ $\forall x \in I$.

② If 'f' is twice Differentiable function on an interval I then 'f' is concave function if and only if $f''(x) \leq 0 \ \forall \ x \in I$ or $f''(x) < 0$ $\forall \ x \in I$

③ The points of the curve at which it's changing i.e. convexity and concavity are points of inflexion.

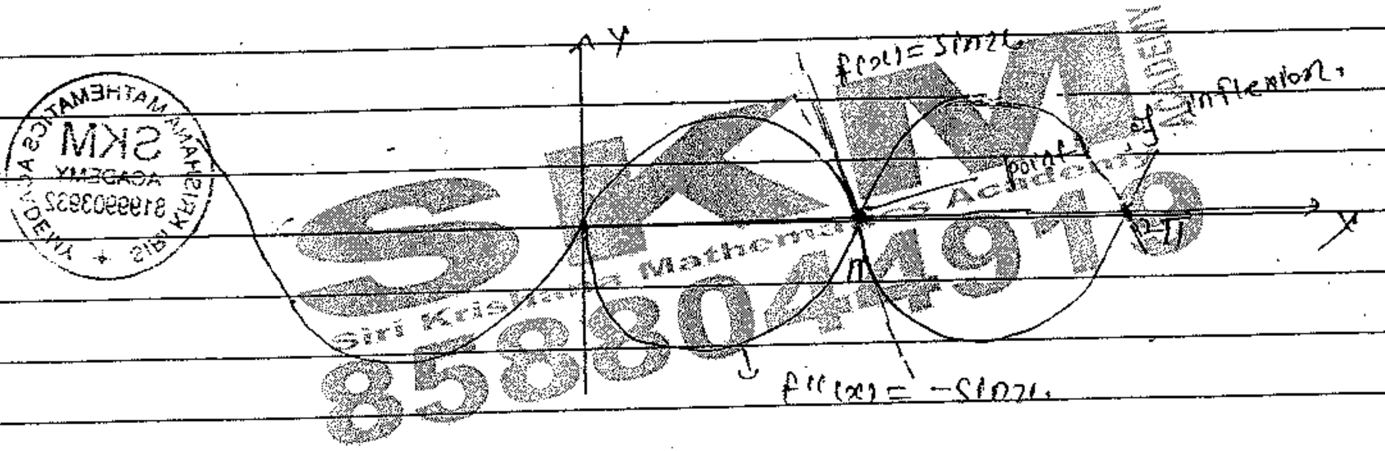
i.e. if 'f' is twice differentiable and $f''(x) = 0$ then x is called the point of inflexion.

Eg:- $f(x) = \sin x$.

$f''(x) = -\sin x$.

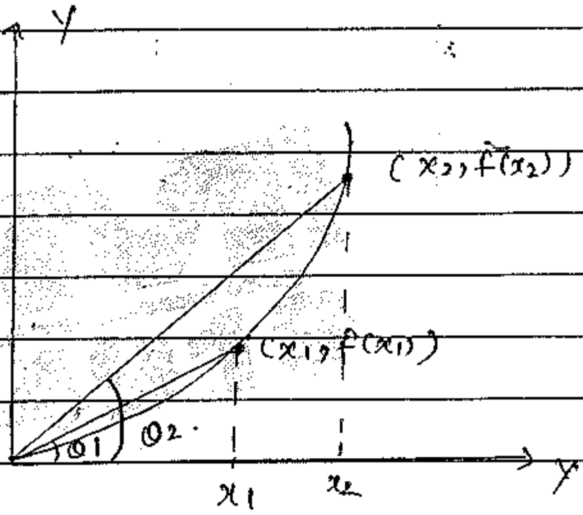
$f''(x) = 0 \Rightarrow -\sin x = 0$

$x = n\pi \ ; \ n \in \mathbb{Z}$.



In graph of sin x in $[0, 2\pi]$, at every point $x \in [0, \pi]$ tangent above the curve so concave but in $[\pi, 2\pi]$ tangent lies below the curve so $x = \pi$ is point of inflexion where concavity and convexity going to be changed.

Q4 $f(x)$ be a convex function and $f(0)=0$ then $g(x) = \frac{f(x)}{x}$, $x \in (0,\infty)$ is increasing function.



Here $\theta_1 < \theta_2$

$\tan\theta_1 < \tan\theta_2$

$\frac{f(x_1)}{x_1} < \frac{f(x_2)}{x_2}$

$g(x_1) < g(x_2)$

$\Rightarrow g(x_1) < g(x_2)$ whenever $x_1 < x_2$

$\Rightarrow g(x)$ is ↑ function.

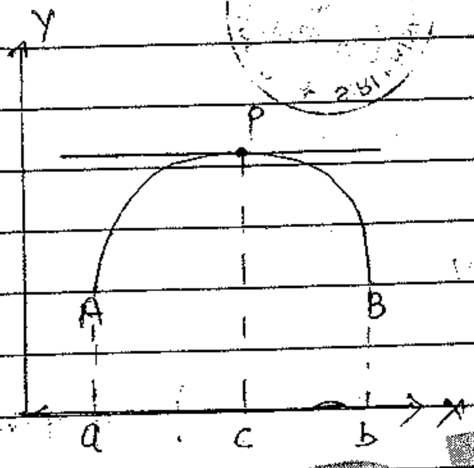
Rolle's Theorem:- Let 'f' is cts function on $[a,b]$ and differentiable on (a,b) and $f(a)=f(b)$ then $\exists\ c \in (a,b)$ s.t $f'(c)=0$

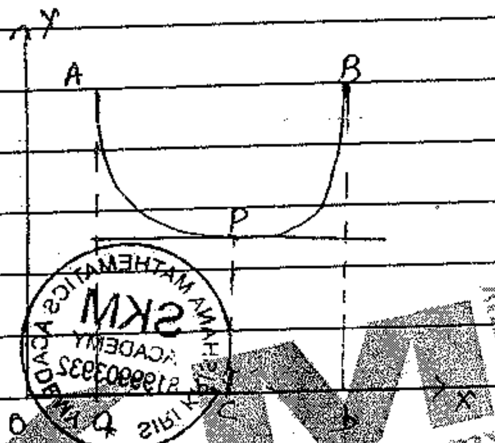
Geometrical Interpretation of Rolle's Theorem:-

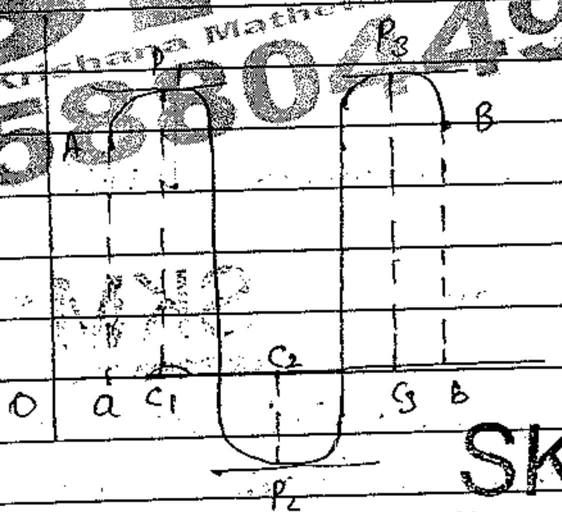
Let A and B are points on the graph of $y=f(x)$ corresponding $x=a$ and $x=b$.

$f(x)$ is continuous on $[a,b]$ $\Rightarrow$ graph of $f(x)$ is without gap or jumps from A to B and $f(x)$ is derivable on (a,b) $\Rightarrow$ the graph $f(x)$ has a tangent at every point between A & B. $f(a)=f(b) \Rightarrow$ points A and B are equidistance from x-axis.

Now possible graphs of $f(x)$ according to all 3 conditions.







from figures it is clear that ∃ atleast one point P on the curve between A and B where tangent is parallel to the x-axis.

$\Rightarrow$ Slope of tangent at P (or P_1, P_2 & P_3) = 0

$\Rightarrow$ $f'(x) = 0$ at P (or P_1, P_2, P_3)

If 'c' is abscissa of P, then $f'(c) = 0$, $a<c<b$

Hence, if all 3 conditions satisfied then Rolle's thm. confirms atleast one point $c \in (a,b)$ where the tangent is ∥le to x-axis.

Note:- Rolle's Thm is not applicable to the function if f(x) fails one or two or three conditions, which are given in Rolle's thm.

Note !- converse of Rollel's thm is Not true i'e. $f'(x)$ may vanishes at a point $c \in (a,b)$ without $f(x)$ satisfying all thee conditions of Rollel's ~~th~~ thm.

Eg:- $f(x) = \begin{cases} 0 & \text{when } 0 < x \le 1 \\ x+1 & \text{when } 1 < x \le 2 \end{cases}$

Here $f(x)$ is Not continuous and diff. at $x=1$

i'e $f(x)$ is Not ctc and diff on $[0,2]$. and $(0,2)$ Respectively. Also $f(0) \ne f(2)$.

But $f'(x)=0 \quad \forall \; x \in (0,1) \subset (0,2)$

Hence, condition are only sufficient Not Necessary.

Q. Verify the Rollel's Thm for the function $f(x) = \sqrt{1-x^2}$ in $[-1,1]$.

Soln:- $f(x)$ is ctc in $[-1,1]$

$f'(x) = \frac{-x}{\sqrt{1-x^2}}$ exist for $\forall \; x \in (-1,1)$.

$\Rightarrow f(x)$ is derivable in $(-1,1)$.

but Not derivable at $x = \pm 1$.

Also $f(-1) = 0 = f(1)$

All 3 conditions are satisfied of Rollel's Thm Hence $\exists \; c \in (-1,1)$ s.t $f'(c) = 0 \Rightarrow \frac{-c}{1-c^2} = 0 \Rightarrow c = 0$.

Q. The equation $x^3 - 3x + k = 0$ has No two distinct Roots in $[0,1]$ for any value of k.

Sol^n:- Let ∃ k' sot x^3-3x+k' has 2 distinct Roots α, β in $[0,1]$.

W.L.O.G. $\alpha > \beta$ & $\alpha \neq \beta$.

and $\alpha, \beta \in [0,1]$ so $0 < \alpha < 1$ & $0 < \beta < 1$

Now α, β are Roots of $x^3-3x-k'=0$ so

$$\alpha^3-3\alpha-k'=0$$

$$\beta^3-3\beta-k'=0$$

i.e $f(\alpha)=f(\beta)=0$

where $f(x)=x^3-3x+k'$

$f(x)$ is Poly. So $f(x)$ is cts on $[0,1]$ and diff on $(0,1)$ and $f(\alpha)=f(\beta)=0$. then ∃ $c \in (\alpha, \beta)$ sot $f'(c)=0$

$$f'(x)=3x^2-3=0$$

$$f'(c)=0$$

$$\Rightarrow 3c^2-3=0$$

$$\Rightarrow c=\pm 1 \text{ (contradicts)}$$

Because $c \in (\alpha, \beta)$ and $\alpha, \beta \in [0,1]$. and $0<\alpha<1$ and $0<\beta<1$ so $c \notin (\alpha, \beta)$

So there doesn't exist k for which $f(x)$ has 2 distinct Roots in $[0,1]$.

Result :- if $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_n}{n-1}+\dots+\frac{a_{n-1}}{2}+a_n=0$

then the equation $a_0x^n+a_1x^{n-1}+\dots+a_n=0$ has atleast one Root b/w 0 and 1

Result :- If f is Non-Negative differentiable function then every zero of f is also zero of f'

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Result: If f is Non-Negative thrice differentiable function on (0,1) has two distinct zero in (0,1) then ∃ $c \in (0,1)$ s.t. $f'''(c) = 0$.

Note: $f(0) = f(1) = f'(1) = --- = f^{n-1}(1) = 0$ and f is continuously differentiable then ∃ $c \in (0,1)$ s.t. $f^{n}(c) = 0$

NET
Qu Let $I = \{1\} \cup \{2\} \subseteq \mathbb{R}$ for $x \in \mathbb{R}$ let $\phi(x) = \text{dist}(x, I) = \text{Inf}\{|x-y| ; y \in I\}$ then —

Soln: $\phi(x) = \text{Inf}\{|x-y| ; y \in I\}$

$$\phi(x) = \begin{cases} |x-1| & ; \ x \leq 3/2 \\ |x-2| & ; \ x > 3/2 \end{cases}$$

f(x) is cts on ℝ and Not Diff only at $x = 1, \frac{3}{2}, 2$.

NET
Qu
$$f(x) = \begin{cases} x \cos \frac{\pi}{2x} & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$$

f(x) is Not cts on ℝ but Not diff at 0.

Lagrange's Mean Value theorem:-

If 'f' be a function defined on [a,b] and 'f' is continuous on closed interval [a,b] and derivable on open interval (a,b) and $f(a) \neq f(b)$ then there exist atleast one Real Number $c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

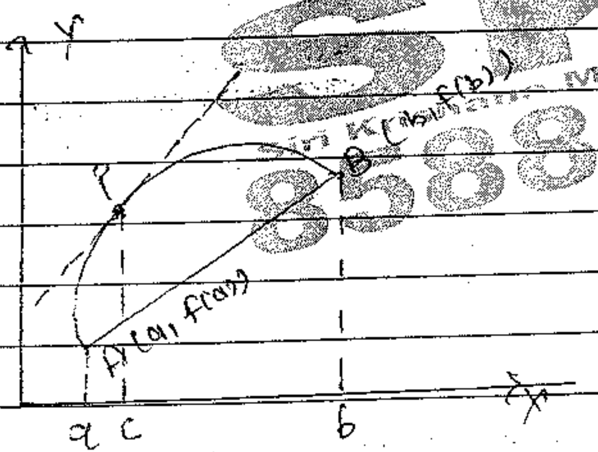
Note:- when $f(a)=f(b)$ then L.M.V.T reduces to Rolle's Theorem.

Geometrical Interpretation of L.M.V.T:-

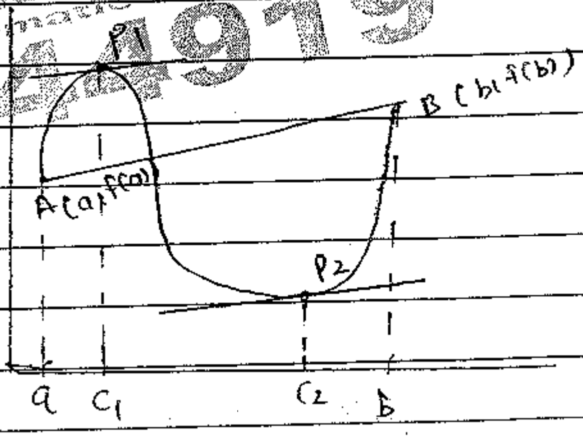
Let A and B two points on the graph of $y=f(x)$ corresponding to $x=a$ & $x=b$.

$f(x)$ is cts on $[a,b]$ ⇒ graph of $f(x)$ has No jump from A to B.

$f(x)$ is diff on (a,b) ⇒ at every point in (a,b), the curve has unique tangent.



(1)



(2)

from fig (1) & (2) we can say ∃ one point b/w

A & B the tangent at which Parallel to Chord AB so L.M.V.T confirms atleast one point in (a,b) at which tangent is ||el to chord AB

if 'c' is abscissa of P (or P_1, P_2), then the slope of tangent at P is f'(c).

∵ chord AB is ||el to the tangent at P

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = f'(c).$$

Result :- If f is cts on [a,b] and derivable on (a,b) then $f'(x)=0 \ \forall \ x \in [a,b] \Rightarrow f$ is constant on [a,b].

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② If 'f' and 'g' are (i) cts on [a,b]
(ii) derivable on (a,b)
then $f'(x) = g'(x) \ \forall \ x \in [a,b]$ then
f-g is constant on [a,b].

Que verify the L.M.V.T for function $f(x) = x^2-3x+2$ in [-2,3].

$f(x) = x^2-3x+2$; $f(a) = f(-2) = 12$
$f(b) = f(3) = 2$
$f'(x) = 2x-3$

f(x) is Poly so cts on [-2, 3] and derivable on (-2,3) then ∃ c ∈ (-2,3) s.t.

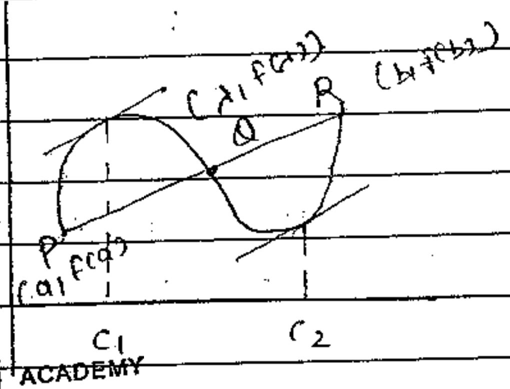
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{2-12}{3-(-2)} = 2c-3 \Rightarrow 10c-15=-10$$

$$\Rightarrow c = 1/2$$

which is clearly within $(-2,3)$. Hence L.M.V.T is verified.

Que If $y = f(x)$ is twice diff on (a,b) and cts on $[a,b]$ if the line segment $(a,f(a))$ & $(b,f(b))$ intersect the curve $y=f(x)$ at third point then show that $c \in (a,b)$ s.t $f''(c)=0$



Slope of PQ = $\dfrac{f(\lambda)-f(a)}{\lambda-a} = f'(c_1)$

Slope of QR = $\dfrac{f(b)-f(\lambda)}{b-\lambda} = f'(c_2)$

By L.M.V.T

but P, Q, R are collinear so $f'(c_1) = f'(c_2)$

So $f(x)$ is cts & diff and $f'(c_1) = f'(c_2)$ then by Rolle's Thm $\exists\ c \in (c_1,c_2)$ s.t $f''(c)=0$

Hence $(c_1,c_2) \subset (a,b)$ so $\exists\ c \in (a,b)$ s.t $f''(c)=0$.

Standard form of L.M.V.T :

$$\boxed{f(a+h) = f(a) + h \cdot f'(a+\theta h).}$$

Let 'f' be a cts on $[a,b]$ and diff on (a,b) then by L.M.V.T $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ for some

$c \in (a,b)$.

if $b-a=h \Rightarrow b=a+h$ & $c=a+\theta h$; $0<\theta<1$

Now $\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$

$$\Rightarrow \boxed{f(a+h)=f(a)+h\,f'(a+\theta h)} \text{ ——— (1)}$$

$f(a+h)= f(a)+h\,f'(a+\theta h)$

$f(a+h)= f(a)+h\,f'(a+\theta h)+\frac{h^2}{2!}f''(a+\theta h)$

$f(a+h)= f(a)+h\,f'(a+\theta h)+\frac{h^2}{2!}f''(a+\theta h)+\frac{h^3}{3!}f'''(a+\theta h)$

$\vdots$

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$f(a+h)= f(a)+h\,f'(a+\theta h)+ - - - +\frac{h^n}{n!}f^{(n)}(a+\theta h)$

Here $\frac{h^n}{n!}f^{(n)}(a+\theta h)$ called Error term.

Let $a+h=x \Rightarrow h=x-a$.

$$\boxed{\begin{aligned} f(x) &= f(a)+(x-a)\,f'(a+\theta h)+\frac{(x-a)^2}{2!}f''(a+\theta h)+ - - \\ &\quad - - - + \frac{(x-a)^n}{n!}f^n(a+\theta h) \text{ ——— ②} \end{aligned}}$$

Here (1) is Standard form of L.M.V.T and (2) is called taylor's Expension of f(x).

Note :- $f(x) = x^m \sin\frac{1}{x}$ then f(x) is diff up to $[\frac{m}{2}]$ times.

Eg :- $f(x) = x^5 \sin\frac{1}{x}$ is diff up to $[\frac{5}{2}] = [2.5] = 2$ times.

Note :- $f(x) = \begin{cases} x^n \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

f(x) is diff $[\frac{n}{2}]$ times and

$$f^{[\frac{n}{2}]}(x) - \begin{cases} \rightarrow \text{continuous if } n \text{ is odd.} \\ \rightarrow \text{discontinuous if } n \text{ is even.} \end{cases}$$

# if $n = 5$ then $f^{[\frac{5}{2}]}(x) = f''(x)$ is cts.

But if $n = 4$ then $f^{[\frac{4}{2}]}(x) = f''(x)$ is Not cts at '0'.

# Cauchy Mean Value theorem :-

Let f and g are cts on [a,b] and diff on (a,b) then ∃ $c \in (a,b)$ st

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad g'(x) \neq 0 \text{ in } (a,b).$$

or $g'(c)(f(b)-f(a)) = f'(c)(g(b)-g(a))$

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Results:-

(i) $|x|^n$; $n=1$ Differentiable excep 0

$|x|^2$; $n=2$ Differentiable.

· $|x|^n$; $n>1$ is Differentiable everywhere.

(ii) $|x+\alpha|^n$; $\alpha>0$; $n>1$, $n \in \mathbb{N}$ is differentiable every where.

(iii) $\sin|x+\alpha|$; $x=-\alpha$ Not diff - $\alpha>0$.

iv) $|\sin x|+2$; $x=n\pi$ Not diff.

v) $\left.\begin{array}{l}\cos|x| \\ \cos|x+\alpha|\end{array}\right\}$ is diff everywhere but

$|\cos x|$ Not diff at $(2n+1)\pi/2$.

vi) e^x diff through out.

vii) $e^{|x|}$ diff everywhere except $x=0$

viii) $e^{|x+\alpha|}$ at $x=-\alpha$; $\alpha>0$ is Not Differentiable.

ix) $|\log x|$ at $x=1$ Not diff.

x) $\log|x|$ diff in its Domain

xi) $|\tan x|$ Not diff at $n\pi$.

Results:- Between two consecutive Root of f ∃ atleast one Root of its Derivative.

Result:- Between two Roots of Polynomial ∃ at least one Root of its Derivative.

Ex:- $f(x)=\sin x$ in $[0,\pi]$.

$f'(x)=\cos x$.

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$f'(\pi/2) = \cos\frac{\pi}{2} = 0.$

(*) Let I be interval, function is $f: I \longrightarrow \mathbb{R}$ if f is diff and bdd then f is satisfy Lipscitz condition then function is U.C.

(*) Let I be interval $I = (a,b)$, c is the interior point of I, function has local Extremum then $f'(c)$ exist or $f'(c)$ does not exist.

Note $f'(c)$ = exist $\Rightarrow f'(c) = 0$.

Ex:- $f(x) = \sin x$, $[0,\pi]$.

$f'(x) = \cos x$

$f'(\pi/2) = 0$ local Minimum.

$f'(0) = 1$ local Maximum.

$f'(c)$ exist.

Ex:- $f(x) = |x|$ in $[-1,1]$

$f'(0)$ = does Not exist But local Minimum at 0.

First derivative test :-

a) Let $I = (a,b)$ interval $f: I \longrightarrow \mathbb{R}$ & f is diff $(a,c]$ and $[c,b)$ $\exists$ a δ-Nbd $(c-\delta, c)$, $f'(x) > 0$ $\forall x \in (c-\delta, c)$ and $f'(x) < 0$ $\forall x \in (c, c+\delta)$ then f has local Maxima at c.

iii) ∃ a δ Nbd $(c-\delta, c+\delta)$ $f'(x) > 0$ ∀ $x \in (c-\delta, c)$ and $f'(x) > 0$ ∀ $x \in (c, c+\delta)$ if $f'(x)$ keeps the same sign then function has No extremum.

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$f(x) = x$

$f'(x) = 1$

$f'(c) = 1$

keeps same sign No local Max. or Minima.

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#! Nth derivative test :-

Let I be Interval $f : I \longrightarrow \mathbb{R}$. c is interior point of I, $f : I \longrightarrow \mathbb{R}$, $c \in I^{\circ}$.
$f'(c) = f''(c)$ ____ $f^{n-1}(c) = 0$ and $f^{n}(c) \neq 0$

i) if n is odd then function is No Extremum.
ii) if n is even then function has local Max. if $f^{n}(c) < 0$ and f has local Min if $f^{n}(c) > 0$.

Ex:- $f(x) = x^3 - x$ in $[-1, 1]$

$f'(x) = 3x^2 - 1$

Put $f'(x) = 0$

$3x^2 - 1 = 0$

$x = \pm 1/\sqrt{3}$

$f''(x) = 6x.$

$f''\left(\frac{1}{\sqrt{3}}\right) = 6 \times 1/\sqrt{3} > 0$ Minima.

and $f''\left(-\frac{1}{\sqrt{3}}\right) = -6 \cdot \frac{1}{\sqrt{3}} < 0$ Maxima

$\Rightarrow x = \pm \frac{1}{\sqrt{3}}$ is point of extremum.

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The RIEMANN INTEGRAL.

Partition of closed interval :-

Let I = [a,b] be a given closed interval. Then ∃ finite set P of points $x_0, x_1, x_2, \ldots, x_n$ s.t. $a = x_0 \le x_1 \le x_2 \ldots \le x_{n-1} \le x_n = b$

i.e $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ s.t $x_i \le x_{i+1}$ ∀ i = 1 to n then P is called a Partion of [a,b]. The (n+1) points $x_0, x_1, x_2, \ldots, x_n$ are called partition points of P. Partition of [a,b] denoted by P[a,b]. The n closed sub-intervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2]$ ... $I_r = [x_{r-1}, x_r]$ ... $I_n = [x_{n-1}, x_n]$. determin by P are called Segments of the Partition P.

Note :- P be a set s.t $P \subseteq [a,b]$

Length of each interval say ith interval
$\Delta x_i = x_i - x_{i-1}$

Note :- Partition is also known as dissection or Net.

Note :- By changing the partition points, partition can be changed and Hence, there can be infinite Number of Partitions of the interval.

Norm of Partition :- The Maximum length of Sub-intervals of a Partition P is called the Norm or

Mesh of the Partition P and denoted by $||P||$ or $\mu(P)$.

Thus; $||P|| = \mu(P) = \text{Max.} \{\Delta x_1, \Delta x_2, ---, \Delta x_r, --- \Delta x_n\}$

or

$$\mu(P) = \text{Max} \{x_r - x_{r-1}, r = 1, 2, --- n\}$$

Note :- $\sum_{r=1}^{n} \Delta x_r = b - a$

Refinement of a Partition :- Let P and P^* are two Partions of $[a,b]$ s.t. $P \subset P^*$ then P^* is called Refinement of Partition of P on $[a,b]$. we also say P^* finer than P.

Note :- if P^* is finear than P, then every point of P is used in the construction of P^* and P^* have atleast one additional point.

i.e. $P = \{a = x_0, x_1, ---, x_n = b\}$ then $\exists\ t \in P^*$ s.t. $x_{r-1} < t < x_r$.

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Note :- $\mu(P^*) \leq \mu(P)$

Result :- Let 'f' be a bounded function on $[a,b]$ and Let $M = \sup f(x)$, $x \in [a,b]$ and $m = \text{Inf} f(x)$; $x \in [a,b]$ i.e $m \leq f(x) \leq M \ \forall \ x \in [a,b]$

Now if m_r is the infimum of $f(x)$ in rth interval.

and M_r is the Supremum of f(x) in rth interval then

$$\boxed{m \le m_r \le M_r \le M}$$

Upper and lower Darboux Sum :-

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function and $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ be a Partition of $[a,b]$. 'f' is Bounded on $[a,b]$, 'f' is also Bounded on each sub-interval. So M and m are ~~supr~~ supremum and infimum of f(x) in $[a,b]$. And M_r and m_r are called Supremum and infimum of f(x) in rth interval $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots$ then

$$\sum_{r=1}^{n} M_r \Delta x_r = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$$ is

called upper Darboux sum and,

$$\sum_{r=1}^{n} m_r \cdot \Delta x_r = m_1 \cdot \Delta x_1 + m_2 \Delta x_2 + \cdots + m_r \Delta x_n$$ is

called lower Darboux Sum.

upper and lower Darboux sum of function 'f' over the Partion P is denoted by $U(f,P)$ and $L(f,P)$ Respectively.

Note :- Darboux Sums (upper and lower) depend upon the function 'f' and the Partition P.

And this sum always exist for every Bounded function.

$$U(f,P) = M_1\Delta x_1 + M_2\Delta x_2 + \dots + M_r\Delta x_r + \dots + M_n\Delta x_n$$

and

$$L(f,P) = m_1\Delta x_1 + m_2\Delta x_2 + \dots + m_r\Delta x_r + \dots + m_n\Delta x_n$$

Note:- 1) $U(f,P) \geq L(f,P)$

2) $U(f,P^*) \leq U(f,P)$

3) $L(f,P^*) \geq L(f,P) \quad \forall\, P$ on $[a,b]$

Reimann upper and lower Integral:-

Let $U(f,P)$ or $U(P,f)$ and $L(f,P)$ or $L(P,f)$ are upper and lower Darboux Sums Respectively.

Then Infimum of $U(f,P)$ and Supremum of $L(f,P)$ over P are called Reimann upper integral and lower integral Respectively. Denoted by $\overline{\int}_a^b f(x)$ and $\underline{\int}_a^b f(x)\,dx$.

i.e.

$$\operatorname{Inf}_P U(f,P) = \overline{\int}_a^b f(x)\,dx \quad \text{and}$$

$$\operatorname{Sup}_P L(f,P) = \underline{\int}_a^b f(x)\,dx.$$

$\boxed{\text{or}}$

$$\lim_{n\to\infty} \sum_{r=1}^{n} M_r . \Delta x_r = \text{Inf}_P\, U(f,P) = \overline{\int_a^b} f(x)\,dx \quad [\text{upper integral}].$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} m_r . \Delta x_r = \text{Sup}_P\, L(f,P) = \underline{\int_a^b} f(x)\,dx \quad [\text{lower integral}.]$$

Note :- if the sum of upper sums $U(f,P)$ for all posible paritions is bounded below, then upper integral is exist and denoted by $\overline{\int_a^b} f(x)\,dx$.

Note :- If the sum of lower sums $L(f,P)$ for all possibles Partitions is Bounded above, then lower integral is exist and denoted by $\underline{\int_a^b} f(x)\,dx$.

Note :- $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$ may or may Not be equal.

i.e Reimann Upper integral and Reimann lower integral may Not be equal.

Que $f : [0,1] \longrightarrow \mathbb{R}$ st $f(x) = \begin{cases} 1, & x \in Q \cap [0,1] \\ -1, & x \in Q^c \cap [0,1] \end{cases}$

then find the upper sum, lower sum, upper integral and lower integral.

Sol^n:- Let $P = \{0 = x_0, x_1, x_2, \dots, x_n = 1\}$ be a Partition of $[0,1]$.

Now supremum of $f(x) = M_r = 1$ in $[x_{r-1}, x_r]$.
and Infimum of $f(x) = m_r = -1$ in $[x_{r-1}, x_r]$.

So upper sum $U(f,P) = \sum_{r=1}^{n} M_r \cdot \Delta x_r = \sum_{r=1}^{n} 1 \cdot \Delta x_r$

$= \sum_{r=1}^{n} \Delta x_r = 1 - 0 = 1.$

$U(f,P) = 1.$

Lower sum $L(f,P) = \sum_{r=1}^{n} m_r \cdot \Delta x_r = \sum_{r=1}^{n} (-1) \cdot \Delta x_r$

$= -\sum_{r=1}^{n} \Delta x_r = -(1-0) = -1$

$L(f,P) = -1$

Now $\lim_{n\to\infty} \sum_{r=1}^{n} M_r \cdot \Delta x_r = \lim_{n\to\infty} (1) = 1$ (upper integral)

So $\overline{\int_0^1} f(x)dx = 1.$

and $\lim_{n\to\infty} \sum_{r=1}^{n} m_r \cdot \Delta x_r = \lim_{n\to\infty} (-1) = -1$ (lower integral)

So $\underline{\int_0^1} f(x)dx = -1.$

Q $f(x) = \begin{cases} \sin x, & x \in Q \\ 0, & \text{else} \end{cases}$

Find lower integral

Soln:- Infimum of $f(x)$ in $[x_{r-1}, x_r] = m_r = 0$

$L(f,P) = \sum_{r=1}^{n} m_r \cdot \Delta x_r = \sum_{r=1}^{n} 0 \cdot \Delta x_r = 0$

$$\lim_{n\to\infty}\sum_{r=1}^{n} m_r\,\Delta x_r = \lim_{n\to\infty}(0) = 0$$

$\boxed{\text{Lower Integral} = 0}$ ∀ P.

Q4 $f(x) = \begin{cases} \cos x ; \text{ if } x \text{ is Rational in } [0, \pi/4] \\ \sin x, \text{ if } x = 0 \text{ or } x \text{ is irrational in } [0, \pi/4] \end{cases}$

then find $\overline{\int_0^{\pi/4}} f(x)\,dx$ and $\underline{\int_0^{\pi/4}} f(x)\,dx$.

Sol^n:- Let $P = \left\{ 0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \ldots, \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, \ldots, \frac{n\pi}{4n} = \frac{\pi}{4} \right\}$

be any Partition of $[0, \pi/4]$ then for any sub-interval $\left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right]$ Here $\Delta x_r = \frac{\pi}{4n}$

$M_r = \cos\frac{(r-1)\pi}{4n}$ & $m_r = \sin\frac{r\pi}{4n}$ ∵ $\cos x > \sin x$ in $\left[0, \frac{\pi}{4}\right]$

Now $U(f,P) = \sum_{r=1}^{n} M_r \cdot \Delta x_r = \sum_{r=1}^{n} \frac{\pi}{4n} \cos\left(\frac{(r-1)\pi}{4n}\right)$

$$= \frac{\pi}{4n}\left\{ \cos 0 + \cos\frac{\pi}{2n} + \cdots + \cos\frac{(n-1)\pi}{4n} \right\}$$

$$= \frac{\pi}{4n}\left\{ \frac{\cos\left(0 + \frac{(n-1)\pi}{4n}\right) \cdot \sin\left(\frac{n}{2}\cdot\frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{4n}\right)} \right\}$$

$$= \frac{\pi}{4n}\cdot\frac{\cos(n-1)\frac{\pi}{2n}\cdot\sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}$$

So $\lim_{n\to\infty}\sum_{r=1}^{n} M_r\cdot\Delta x_r = \int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\frac{\pi/8n}{\sin\pi/2n}\cdot 2\cos\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\cdot\sin\frac{\pi}{8}\right]$

$$= 1\times 2\cos\frac{\pi}{8}\cdot\sin\frac{\pi}{8}$$

$$= \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

So $\int_0^{\pi/4} f(x)\,dx = \frac{1}{\sqrt{2}}$ Ans

Now $L(f,P) = \sum_{r=1}^{n} m_r\cdot\Delta x_r = \sum_{r=1}^{n}\frac{\pi}{4n}\cdot\sin\frac{(r-1)\pi}{4n}$

$$= \frac{\pi}{4n}\left\{\sin 0 + \sin\frac{\pi}{4n} + \cdots + \sin\frac{(n-1)\pi}{4n}\right\}$$

$$\frac{\pi}{4n}\left\{\frac{\sin\left(0+\frac{(n-1)}{2}\cdot\frac{\pi}{4n}\right)\cdot\sin\left(\frac{n}{2}\cdot\frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2}\cdot\frac{\pi}{4n}\right)}\right\}$$

$$= \frac{\pi}{4n}\cdot\frac{\sin(n-1)\frac{\pi}{8n}\cdot\sin\frac{\pi}{8}}{\sin\pi/8n}$$

So $\int_0^{\pi/4} f(x)\,dx = \lim_{n\to\infty}\sum_{r=1}^{n} M_r\,\Delta x_r = \lim_{n\to\infty}\left[\frac{\pi/8n}{\sin\pi/8n}\cdot 2\sin\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\cdot\sin\frac{\pi}{8}\right]$

$$= 1\times 2\sin^2\frac{\pi}{8} = 1-\cos\frac{\pi}{4} = 1-\frac{1}{\sqrt{2}}$$

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So. $\int_{0}^{\pi/4} f(x)\,dx = 1 - \frac{1}{\sqrt{2}}$ Ans

Reimann Integrable :- if 'f' is a bounded function on closed interval [a,b] then 'f' is said to be integrable (or Reimann Integrable) on [a,b]. if upper and lower Reimann integral are same

i.e $\overline{\int_{a}^{b}} f(x)\,dx = \underline{\int_{a}^{b}} f(x)\,dx$

and denoted by R[a,b] i.e f ∈ R[a,b] or

$\int_{a}^{b} f(x)\,dx$ and called Reimann integral of 'f' with Repect to x in [a,b].

Eg:- $f(x) = x$; $x \in [0,1]$.

let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots \frac{r-1}{n}, \frac{r}{n}, \cdots \frac{n}{n} = 1\right\}$ be a

Partition of [0,1]. Here $\Delta x_r = \frac{1}{n}$

$m_r = $ Inf. of $f(x)$ in $[x_{r-1}, x_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right] = \frac{r-1}{n}$

$M_r = $ Sup. of $f(x)$ in $[x_{r-1}, x_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right] = \frac{r}{n}$

$L(f,P) = \sum_{1}^{n}\left(\frac{r-1}{n}\right)\frac{1}{n} = \frac{1}{n^2}\sum_{1}^{n}(r-1) = \frac{n(n-1)}{2n^2}$

$U(f,P) = \sum_{1}^{n}\left(\frac{r}{n}\right)\left(\frac{1}{n}\right) = \frac{1}{n^2}\sum_{1}^{n} r = \frac{n(n+1)}{2n^2}$

$$\overline{\int_0^1} f(x)dx = \lim_{n\to\infty} U(f,P) = \lim_{n\to\infty}\left(\frac{n(n+1)}{2n^2}\right) = \frac{1}{2}$$

$$\underline{\int_0^1} f(x)dx = \lim_{n\to\infty} L(f,P) = \lim_{n\to\infty}\left(\frac{n(n-1)}{2n^2}\right) = \frac{1}{2}$$

Here $\overline{\int_0^1} f(x)dx = \underline{\int_0^1} f(x)dx$

So $f \in R[0,1]$.

Note :- for showing 'f' is Reimann integrable, we have to take partition of closed interval $[a,b]$ into n-subinterval. If partition P has only some points like 2, 3 or 4, — — then it will Not be R.I. as we can't say about 'f' is R.I or Not.

Eg :- $f(x) = x$ on $[0,1]$.

let $P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$ be a Partition of $[0,1]$

with 5 points. Here $\Delta x = \frac{1}{4}\ \forall\ i$

Subintervals $I_1 = \left[0, \frac{1}{4}\right]$, $I_2 = \left[\frac{1}{4}, \frac{2}{4}\right]$, $I_3 = \left[\frac{2}{4}, \frac{3}{4}\right]$

, $I_4 = \left[\frac{3}{4}, 1\right]$

Then $M_1 = \frac{1}{4}$, $M_2 = \frac{2}{4}$, $M_3 = \frac{3}{4}$, $M_4 = 1$ and

$m_1 = 0$, $m_2 = \frac{1}{4}$, $m_3 = \frac{2}{4}$, $m_4 = \frac{3}{4}$

So $U(f,P) = \sum_{r=1}^{4} M_r \cdot \Delta x_r = \frac{1}{4}\cdot\frac{1}{4} + \frac{2}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + 1\cdot\frac{1}{4}$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} = \frac{5}{8}$$

& $L(f,P) = \sum_{r=1}^{4} m_r \cdot \Delta x_r = 0\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{2}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{8}$.

Now $\lim_{n\to\infty} \sum_{r=1}^{4} M_r\, \Delta x_r = \overline{\int_0^1} f(x)\,dx = \frac{5}{8}$

& $\lim_{n\to\infty} \sum_{r=1}^{4} m_r \cdot \Delta x_r = \underline{\int_0^1} f(x)\,dx = \frac{3}{8}$

Here $\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx$

So $f \notin R[0,1]$ with Partition P.

So finally we can't write $f \notin R[a,b]$. The Meaning of R- integrable is only the upper integral = lower integral.

i.e. we always take $\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$

And for determing $\overline{\int_a^b} f(x)dx$ or $\underline{\int_a^b} f(x)dx$ we take $n \to \infty$, it means when each sub interval's length approching to zero then if $\overline{\int_a^b} f(x)dx$ and $\underline{\int_a^b} f(x)dx$ exist and equal then we say f is R- Integrable otherwise Not.

Results :

if $f : [a,b] \longrightarrow \mathbb{R}$ is a bounded function and P be Partition of $[a,b]$ and P^* is Refinement of P then

1) $m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$
2) $L(-f,P) = -U(f,P)$ and $U(-f,P) = -L(f,P)$
3) $U(f+g,P) \leq U(f,P) + U(g,P)$
4) $L(f+g,P) \geq L(f,P) + L(g,P)$
5) $L(f,P) \leq L(f,P^*) \leq U(f,P^*) \leq U(f,P)$
6) $U(f,P) - L(f,P) \geq U(f,P^*) - L(f,P^*)$.

Result:- If P^* is Refinement of P containing p points More than P and $|f(x)| \leq k \;\forall\; x \in [a,b]$ then

1) $L(f,P) \leq L(f,P^*) \leq L(f,P) + 2pk\delta$
2) $U(f,P) \geq U(f,P^*) \geq U(f,P) - 2pk\delta$

where $\|P\| = \delta$.

Result:- If $P_1, P_2 \in P[a,b]$ then

1) $L(f,P_1) \leq U(f,P_2)$
2) $L(f,P_2) \leq U(f,P_1)$

i.e No lower Sum can exceed any upper Sum.

Oscillatory Sum:- Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function and Let $P = \{a = x_0, x_1, \ldots\ldots x_n = b\}$ be a Partition of $[a,b]$. M_r and m_r are the Supremum and infimum of 'f' in r^{th} interval $[x_{r-1}, x_r]$, $r = 1,2,\ldots\ldots n$ then

$$U(f,P) - L(f,P) = \sum_{r=1}^{n} M_r \Delta x_r - \sum_{r=1}^{n} m_r \Delta x_r$$

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$U(f,P) - L(f,P) = \sum_{r=1}^{n} (M_r - m_r)\cdot \Delta x_r$ is called oscillatory sum of 'f' corresponding to the Partitio P and is denoted by $\omega(f,P)$.
Here $(M_r - m_r)$ denotes the oscillation of 'f' is rth interval.

Note:- Each oscillatory sum consists of a finite Number of Non-Negative terms.

i.e $M_r - m_r \ge 0 \ \forall \ r = 1, 2, - - n$.
then $\omega(f,P) \ge 0 \ \forall \ P$.

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Results:- 1) $\omega(f,P) \ge \omega(f,P^*)$
2) $\omega(f+g,P) \le \omega(f,P) + \omega(g,P)$
3) $\omega(f,P) - \omega(f,P^*) \le 4pk\delta$

Here $|f(x)| \le k$ and p is points which are in P^* more than P and $\|P\|$

Result:- As we know $m \le m_r \le M_r \le M$
multiplying by Δx_r or $(b-a)$ in each, then

$m\cdot\Delta x_r \le m_r\cdot\Delta x_r \le M_r\cdot\Delta x_r \le M\cdot\Delta x_r$
taking Σ over each, then

$$\sum m\cdot\Delta x_r \le \sum_1^n m_r\Delta x_r \le \sum_1^n M_r\cdot\Delta x_r \le \sum M\cdot\Delta x_r \quad \text{—(i)}$$

$$\sum m\cdot\Delta x_r \le L(f,P) \le U(f,P) \le \sum M\cdot\Delta x_r$$

when $n \to \infty$

$$m \cdot \sum \Delta x_r \le \lim_{n\to\infty} \sum_{1}^{n} m_r \cdot \Delta x_r \le \lim_{n\to\infty} M_r \cdot \Delta x_r \le M \cdot \sum \Delta x_r$$

$$m(b-a) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le M(b-a)$$

Here $\Delta x_r = (b-a)$

Condition for Integrability :-

A Necessary and Sufficient condition for the integrability of a bounded function 'f' is that to every $\epsilon > 0$, there exist a $\delta > 0$ s.t $\forall$ Partition P of $[a,b]$

$\boxed{U(f,P) - L(f,P) < \epsilon}$ whenever $\mu(P) < \delta$ or $\|P\| < \delta$.

(or)

A Necessary and Sufficient condition for the integrability of a Bounded function f is that

$\lim \{U(f,P) - L(f,P)\} = 0$, where the Norm $\mu(P)$ or $\|P\|$ of Partition P tends to zero

Result :- $\sum (M_r - m_r)\Delta x_r = U(f,P) - L(f,P)$

The L.H. side is called oscillatory sum of 'f' over P then $f \in R[a,b]$ iff any $\epsilon > 0 \ \exists \ \delta > 0$ s.t

$$\sum (M_r - m_r)\Delta x_r < \epsilon \text{ whenever } \mu(P) < \delta.$$

i.e Oscillatory Sum can be made as small as we do if 'f' is R.I. when $\mu(P) < \delta$.

Result:- if $f \in R[a,b]$ then upper and lower integral are same and give the area bound by $f(x)$, $x = a$ and $x = b$.

Result:- $\sum f(t_r)\, \Delta x_r$ exists finitely then $f \in R[a,b]$ & $\lim \mu(P) \to 0$

where $t_r \in [x_{r-1}, x_r]$.

Hence $$\lim_{\|P\| \to 0} \sum f(t_r)\, \Delta x_r = \int_a^b f(x)\,dx = \text{Actual area}$$

Result:- If 'f' is bounded and integrable on $[a,b]$ then there exist a Number λ lying b/w the bounds of 'f' s.t

$$\int_a^b f(x)\,dx = \lambda \cdot (b-a)$$

Result:- if 'f' is cts on $[a,b]$ and $f \in R[a,b]$ then $\exists$ 'c' b/w a and b s.t

$$\int_a^b f(x)\,dx = (b-a) \cdot f(c)$$

Result:- if 'f' is bdd and integrable on $[a,b]$ and K is Number s.t $|f(x)| \le K \ \forall\ x \in [a,b]$

then $$\left| \int_a^b f(x)\,dx \right| < K\,|b-a|$$

Result- if f is bounded and integrable on $[a,b]$, and $f(x) \ge 0 \ \forall\ x \in [a,b]$ then

$\int_a^b f(x) \geq 0$ if $b \geq a$ and $\int_a^b f(x)dx \leq 0$ if $b \leq a$.

Result:- If 'f' and 'g' are bounded and integrable on $[a,b]$ s.t $f \geq g$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx \text{ if } b \geq a$$

and $$\int_a^b f(x)dx \leq \int_a^b g(x)dx \text{ if } b < a.$$

Darboux Theorem :- If f is bdd function on $[a,b]$ then to every $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

i) $U(f,P) < \overline{\int_a^b} f(x)dx + \epsilon$

ii) $L(f,P) > \underline{\int_a^b} f(x)dx - \epsilon$

For Every Partition P of $[a,b]$ with Norm $\mu(P) < \delta$.

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Algebra of Reimann Integrability :-

If 'f' and 'g' are Reimann integrable on [a,
then.

1> $f+g$ is R- integrable and $\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g($

2> $f-g$ is R- integrable and $\int_a^b (f-g)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

3> $f \cdot g$ is also R - integrable on $[a,b]$.

4> if f/g is bdd on $[a,b]$ then f/g is R-integrable.

5> $|f|$ is R- integrable and $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

6> f^2 is also R - integrable.

7> $\max\{f,g\}$ is R - integrable

8> $\min\{f,g\}$ is R- integrable.

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Result :- A bounded function 'f' is Reimann integrable on $[a,b]$ iff 'f' is bdd and Reimann integrable on $[a,c]$ and $[c,b]$ where c is point of $[a,b]$.

Note:- if f/g is not bdd on $[a,b]$ then f/g is Not R-Integrable on $[a,b]$, where f and g are R-Integrable.

E.g. $f(x) = 1$ in $[0,1]$ and

$$g(x) = \begin{cases} x ; & x \in (0,1] \\ 0 ; & x = 0 \end{cases}$$

then $$\frac{f}{g} = \begin{cases} 1/x & \text{if } x \in (0,1] \\ 0 , & x = 0 \end{cases}$$ is Not R.I

where f/g is Not Bounded on $[0,1]$.

Some theorems which will confirm that given function is R-Integrable:-

1) if 'f' is continuous on $[a,b]$ then $f \in R[a,b]$
Converse Need Not true.

2) If 'f' is Monotonic function on $[a,b]$ then $f \in R[a,b]$, converse Not true.

3) If the set of points of discontinuity of a ~~Bdd~~ Bounded function 'f' on $[a,b]$ is finite then $f \in R[a,b]$.

4) If the set of points of discontinuity of a Bounded function 'f' on $[a,b]$ is C'ble then $f \in R[a,b]$.

5) if the set of points of discontinuity of a Bounded fuction 'f' on $[a,b]$ has finite Number of limit points then $f \in R[a,b]$

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Note:- In each theorem fuction should be Bounded on $[a,b]$.

Note:- Continuity and Monotonicity is the sufficien condition for 'f' to be integrable Not Necessary.

Eg:- $f: [-1,1] \longrightarrow \mathbb{R}$ s.t

$$f(x) = \begin{cases} K > 0 & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

This function has only one point of discontinu So by Theorem 3 f is R-Integrable.

but given 'f' is Not cts and Monotonic on $[a,b$

Que $f: [0,1] \longrightarrow \mathbb{R}$ s.t

$$f(x) = \begin{cases} 1/q & ; x = p/q \text{ in } [0,1] \\ 0 & ; x = 0 \text{ or else} \end{cases}$$

So, $f \in R[0,1]$ because f is Not cts at Rational. So set of discontinuity points is Rational Number which is countable so by Th^m (4) $f \in R[0,1]$.

Que $f: [0,1] \longrightarrow \mathbb{R}$ s.t

$$f(x) = \begin{cases} 1/q & ; x = p/q \\ 1/q^2 & ; x \; \sqrt{p/q} \quad p \,\&\, q \text{ are squar fr} \\ 0 & ; \text{[illegible]} \end{cases}$$

This function $f(x)$ is Not cts at the points of type p/q and $\sqrt{p/q}$. So set of points of discontinuity is c'ble so $f \in \mathbb{R}[0,1]$.

Q. $f(x) = \begin{cases} \frac{1}{2^n} ; & \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \\ 0 ; & x = 0 \end{cases}$ is integrable on $[0,1]$.

$n = 0, 1, 2, \ldots$

$f(x) = 1 \;;\; \frac{1}{2} < x \le \frac{1}{1}$

$f(x) = \frac{1}{2} \;;\; \frac{1}{2^2} < x \le \frac{1}{2}$

$f(x) = \frac{1}{2^2} \;;\; \frac{1}{2^3} < x < \frac{1}{2^2}$

$\vdots$

$f(x) = \frac{1}{2^n} \;;\; \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$ and so on.

Thus we Notice that f is bdd and Monotonic increasing on $[0,1]$ so $f \in R[0,1]$ for $f \in [0,1]$.

And also $f(x)$ cts on $[0,1]$ except at the set of points $\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$, which have only one limit point So by thm (5) $f \in R[0,1]$.

Q. $f(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} ; & x \in (0,1] \\ 0 ; & x = 0 \end{cases}$

Here 'f' has discontinuity at $x=0$ only so $f \in R[0,1]$.

Qu $g(x) = \begin{cases} x^2 \sin 1/x & ; x \in (0,1] \\ 0 & ; x=0 \end{cases}$

$g(x)$ is cts on $[0,1]$ so $f \in R[0,1]$.

Result:- if 'f' is Non-Negative cts function on $[a,b]$ and $\int_a^b f(x)dx = 0$ then $f(x)=0 \ \forall \ x \in [a,b]$.

Result:- f is cts on $[a,b]$ & $f(x) \geq 0 \ \forall \ x \in [a,b$ with $\int_a^b f(x)dx = 0$ then $f \equiv 0$.

Result:- f is cts on $[a,b]$ & $\int_a^b [f(x)]^2 dx \geq 0$ then $f \equiv 0$.

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Result:- If 'f' is Non-Negative Reimann Integrable on $[a,b]$ s.t $\int_a^b f(x)dx = 0$ then at every point of continuity the value of function is zero i.e If 'f' is cts at $x=c \Rightarrow f(c)=0$

Note:- function is Non-Negative and Non-zero then it is quite posible that area is zero.

Eg:- $F(x) = \begin{cases} 1/q & ; x = p/q \text{ in } [0,1] \\ 0 & ; x=0 \text{ or else.} \end{cases}$

Here 'f' is R-integrable and Non-Negative and

and Non-zero.

$$\text{So} \int_a^b f(x)\,dx = \int_a^{\bar{b}} f(x) = \int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} m_r \Delta x_r \quad m_r \in [x_{r-1}, x_r]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} 0\,\Delta x_r \quad \{m_r = 0 \text{ in } [x_{r-1}, x_r]\}$$

$$= 0$$

So ~~lim~~ $\int_a^b f(x)\,dx = 0$

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Q4 Let 'f' and g are defined on $[a,b]$ & bdd if S is subset of $[a,b]$ s.t 'f' and 'g' are agree on $[a,b]-S$ i.e $f(x)=g(x)$ $\forall x \in [a,b]-S$ then.

1> If S is finite and $f \in R[a,b]$ then $g \in$

2> $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$

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Pf:- (i) $h(x) = f(x) - g(x)$ $\forall x \in [a,b]-S$

$h(x) = 0$ $\because$ $(f(x) = g(x)$ on $[a,b]-S]$.

Here $h(x)$ is cts on $[a,b]-S$ so S is the set of points of discontinuity which is finite so $h(x) \in R[x]$ Here $h(x) = f(x) - g(x)$ & $f(x)$ is R-integrable so $g(x)$ is R-integrable

Indefinite Integral :- if $f \in R[a,b]$, then the function defined by

$$F(x) = \int_a^x f(t)\,dt \quad \text{for} \quad x \in [a,b]$$

is called Indefinite Integral of 'f' with base point a [if $c \in [a,b]$ then 'c' can be base point].

Properties :-

1) $F(a) = 0$ 2) $F(b) = \int_a^b f(t)\,dt$

3) F is always continuous on $[a,b]$, Need Not to be differentiable.

Eg :- $f(x) = \begin{cases} -1 & ; -1 < x < 0 \\ 0 & ; x = 0 \\ 1 & ; 0 < x < 1 \end{cases}$

Here $f \in R[a,b]$:

i.e. only one point $x=0$ is point of Discontinuity

So $f \in R[-1,1]$ so $\exists$ F s.t

$$F(x) = \int_{-1}^{x} f(t)\,dt$$

$$F(x) = \begin{cases} -x & ; -1 \le x < 0 \\ x & ; 0 \le x \le 1 \end{cases}$$

Here F(x) is cts on [-1,1] but Not diff at

$x=0$. so $F(x)$ is always continuous but Need Not to be differentiable.

Th^m :- The indefinite integral is continuous on $[a,b]$, infact if $|f(x)| \le M$ ∀ $x \in [a,b]$ then $|F(x)-F(y)| \le M|x-y|$ ∀ $x,y \in [a,b]$.

Conclusion If $f \in R[a,b]$, then its indefinite integral satisfied a Lipschitz condition Here F is continuous and F is uniformely continuous on $[a,b]$.

Th^m :- Let $f \in R[a,b]$ and let f be a continuous on $[a,b]$ or f be a continuous at a point 'c' $\in [a,b]$. Then indefinite integral is differentiable at c and $F(c) = f(c)$.

Conclusion :- If f is cts on $[a,b]$ & $f \in R[a,b]$ then its indefinite F is differentiable on $[a,b]$ and $F'(x) = f(x)$ ∀ $x \in [a,b]$

Note :- If $f \in R[a,b]$ and f is cts on $[a,b]$ then its indefinite integral to be differentiable Not Necessary

Note :- Continuity of f is sufficient condition for the indefinite integral to be Differentiable Not Necessary.

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Eg: $f(x) = \begin{cases} 2x\sin 1/x - \cos 1/x & ; x \neq 0 \text{ in } [-1,1] \\ 0 & ; x = 0 \end{cases}$

and its definite integral

$F(x) = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \text{ in } [-1,1] \\ 0 & ; x = 0 \end{cases}$

is diff. and $F'(x) = f(x)$, whereas 'f' is Not Not cts at $x=0$.

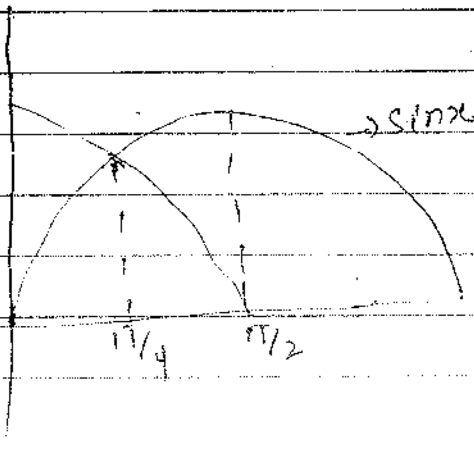
Result:- If 'f' and 'g' $\in R[a,b]$ and continuous on $[a,b]$ and $\int_a^b f(x)dx = \int_a^b g(x)dx$. then

$\exists\ c \in [a,b]$ s.t $f(c) = g(c)$

Result:- If $f \in R[a,b]$ and for $x_1 < x_2$, $F(x_2) \geq F(x_1)$ where $f(x) \geq 0$.

Qu Find upper and lower Reimann integral for given function 'f' defined by.

$f(x) = \begin{cases} \sin x & ; x \in \mathbb{Q} \cap [0, \pi/2] \\ \cos x & ; x \in \mathbb{Q}^c \cap [0, \pi/2] \end{cases}$



$$\overline{\int_0^{\pi/2}} f(x)\,dx = \int_0^{\pi/4} \cos x + \int_{\pi/4}^{\pi/2} \sin x$$

$$= \left[\sin x\right]_0^{\pi/4} + \left[-\cos x\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

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$$\underline{\int_0^{\pi/2}} f(x)\,dx = \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$$

$$= \left[-\cos x\right]_0^{\pi/4} + \left[\sin x\right]_{\pi/4}^{\pi/2}$$

$$= \left[\frac{-1}{\sqrt{2}} + 1\right] + \left[1 - \frac{1}{\sqrt{2}}\right]$$

$$= -\sqrt{2} + 2$$

$$= 2 - \sqrt{2}$$

Hence $\overline{\int_0^{\pi/2}} f(x)\,dx = \sqrt{2}$

and $\underline{\int_0^{\pi/2}} f(x)\,dx = 2 - \sqrt{2}$.

Diff $\Rightarrow$ Cts $\Rightarrow$ R.I $[a,b]$

Defn. Let $I = [a,b]$ closed and bdd interval. $f : I \to \mathbb{R}$ is R.I then by special function then function is cts.

$$f(x) = \begin{cases} 0 \;; & -1 \le x \le 0 \\ 1 \;; & 0 < x \le 1 \end{cases}$$

function discts at $x=0$

$-1 \le x < 0 \Rightarrow \int_{-1}^{x} 0 \cdot dx = 0$

$0 < x \le 1 \Rightarrow \int_{-1}^{x} f = \int_{-1}^{0} f\,dx + \int_{0}^{x} f\,dx$

$= 0 + x$

$f(x) = \begin{cases} 0 & ; -1 \le x \le 0 \\ x & ; 0 < x \le 1 \end{cases}$ special function

is cts

# let $I = [a,b]$ is closed and bdd interval. and f is bdd function and f is cts ~~at~~ $c \in [a,b]$ by special function

$F(x) = \int_{a}^{x} f(t)\,dt$

then function is diff at 'c'

Ex 1 $f(x) = \begin{cases} 0 & ; -1 \le x < 0 \\ x & ; 0 \le x < 1 \end{cases}$

is cts then Prove by special function it is diff.

$-1 \le x < 0 \Rightarrow \int_{-1}^{x} 0\,dx = 0$

$0 < x \le 1 \Rightarrow \int_{-1}^{x} f = \int_{-1}^{0} f + \int_{0}^{x} f$

$= 0 + \frac{x^2}{2}\Big|_{0}^{x}$

$f(x) = \begin{cases} 0 & ; -1 \le x < 0 \\ x^2 & ; 0 \le x \le 1 \end{cases}$ is differentiable.

IMPROPER INTEGRALS :-

Finite and Infinite Intervals :- An Interval is said to be finite or Infinite. Thus the intervals $[a,b]$, $[a,b)$, (a,b), $(a,b]$ are finite with length $(b-a)$ if both a and b are finite.

The intervals $[a,\infty)$, (a,∞), $(-\infty,b)$ and $(-\infty,\infty)$ infinite (or unbdd) Intervals.

Proper Integrals :- The definite integral $\int_a^b f(x)dx$ is called Proper integral if

i) The interval of integration $[a,b]$ is finite (or bdd)
ii) The integral f is bdd on $[a,b]$ i.e. The integrand f is bounded on $[a,b]$.

Improper Integral :- The definite integral $\int_a^b f(x)\,dx$ is said to be an improper integral if either or both the above conditions are Not satisfied. Thus $\int_a^b f(x)dx$ is an Improper Integral, if either the interval of integration $[a,b]$ is Not finite or f is Not bdd on $[a,b]$ or, Neither the Interval $[a,b]$ is finite nor f is bdd over it

E.g:- $\int_1^\infty \frac{dx}{x}$, $\int_{-\infty}^0 e^{2x}dx$, $\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$, $\int_0^1 \frac{1}{x}\,dx$

Second Defination :- Let f is defined on (a,b) then the Integral of the type $I = \int_a^b f(x)dx$ is called improper integral under the following conditions.

i) Either 'a' or 'b' or both at $+\infty$ or $-\infty$.

ii) f has point of infinite discountinuity (P.I.D)
i.e f is unbdd in (a,b).

Note:- We say 'f' has point of Infinite Discontinuity at $x = c$ if $\lim_{x \to c} f(x) = +\infty$ or $-\infty$.

Examples of Improper integrals:-

$\int_0^\infty \sin x^2\,dx$, $\int_0^\infty \frac{\sin x}{x^2}\,dx$, $\int_0^2 \frac{dx}{x-1}$, $\int_{-\infty}^{\infty} x^3\,dx$.

Improper Integral of First Kind:-

A definite integral $\int_a^b f(x)\,dx$ is said to be improper integral of first kind if either a or b or both are infinite so that interval is unbdd. But f is bdd i.e 'f' has no P.I.D in [a,b].

Eg:- $\int_1^\infty \frac{dx}{\sqrt{x}}$, $\int_{-\infty}^0 e^{2x}\,dx$, $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx$.

Improper Integral of Second Kind:-

The Definite integral $\int_a^b f(x)\,dx$ is said to be improper Integral of Second kind if 'a' and 'b' are finite but 'f' has one or more point of infinite Discontinuity. i.e 'f' is Not bdd on $[a,b]$.

Eg:- $\int_0^1 \frac{dx}{x^2}$, $\int_1^2 \frac{dx}{2-x}$, $\int_1^4 \frac{dx}{(x-1)(x-4)}$, $\int_0^2 \frac{dx}{(x-1)^2}$

Improper Integral of Third kind:—

The definite integral $\int_a^b f(x)dx$ is said to be improper integral of third kind if the integrand of 'f' is unbdd i.e a or b or both are Infinite and 'f' is also unbdd on $[a,b]$.

Eg:- $\int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$, $\int_1^\infty \frac{dx}{(x-1)^2}$, $\int_{-\infty}^{0} \frac{1}{(x+2)} dx$.

Improper Integral as the limit of a Proper Integral

1) $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$

2) $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$

3) $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ $\{ c \in (-\infty,\infty) \}$

$= \lim_{t_1\to-\infty} \int_{t_1}^c f(x)dx + \lim_{t_2\to\infty} \int_c^{t_2} f(x)dx$

Note :- The improper integral of Ist kind 1), 2), 3) are said to be convergent if limit on R.H.S exist finitely. And divergent if limit is +∞ or −∞.

④ If $f(x)$ has P.I.D at $x = b$ only then

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to0^+} \int_a^{b-\epsilon} f(x)\,dx$$

⑤ If $f(x)$ has P.I.D at $x = a$ only then.

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to0^-} \int_{a+\epsilon}^b f(x)\,dx.$$

⑥ If $f(x)$ is infinite at $x = c$ only where $a < c < b$

then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\epsilon_1\to0} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to0} \int_{c+\epsilon_2}^b f(x)\,dx$$

Note :- The improper integral of second kind (4), (5), (6) are said to be convergent if limit on R.H.S exist finitely otherwise Divergent.

Note :- In each case if limit on R.H.S is finite then improper integral is convergent and converges to that limit. If limit is +∞ or −∞, then improper integral is Divergent. And if limit

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oscillates then improper integral oscillates.

Examine for convergence the improper integral:—

① $\int_0^\infty e^{-mx}$ $(m>0)$ [First kind.]

$$\int_0^\infty e^{-mx} = \lim_{t\to\infty}\int_0^t e^{-mx}dx = \lim_{t\to\infty}\left[\frac{e^{-mx}}{-m}\right]_0^t = \lim_{t\to\infty}\left[-\frac{e^{-mt}}{m}+\frac{1}{m}\right]$$

$$= \frac{-1}{m}(0-1) = \frac{1}{m} \text{ finite.}$$

So $\int_0^\infty e^{-mx}dx$ is convergent.

② $\int_0^\infty \sin x\,dx$ [First kind].

$$\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx = \lim_{t\to\infty}\left[-\cos x\right]_0^t$$

$$= \lim_{t\to\infty}(1-\cos t)$$

which doesn't exist uniquely. Oscillates b/w 1 & −1

$\Rightarrow \int_0^\infty \sin x\,dx$ oscillates.

③ $\int_0^\infty x\cdot e^{-x^2}dx$ [First kind].

$$\int_0^\infty x\cdot e^{-x^2}dx = \lim_{t\to\infty}\int_0^t x\cdot e^{-x^2}dx$$

Put $x^2=z$

$2x\,dx=dz$

$x\,dx=\frac{dz}{2}$

So. $\int_0^\infty x\cdot e^{-x^2}dx = \lim_{t\to\infty}\int_0^{t^2}\frac{1}{2}e^{-z}dz = \lim_{t\to\infty}\left[\frac{-1}{2}e^{-z}\right]_0^{t^2}$

$= \lim_{t\to\infty}\frac{-1}{2}\left(e^{-t^2}-1\right) = \frac{1}{2}$ (finite).

$\Rightarrow \int_0^\infty x\cdot e^{-x^2}dx$ is convergent and its value is $1/2$.

④ $\int_0^\infty e^{-ax}\cos bx\,dx$ [First Kind].

$\int_0^\infty e^{-ax}\cos bx\,dx = \lim_{t\to\infty}\int_0^t e^{-ax}\cdot\cos bx$

$= \lim_{t\to\infty}\left[\frac{e^{-ax}}{(-a)^2+b^2}(-a\cos bx+b\sin bx)\right]_0^t$

$\because \int e^{ax}\cos bx\,dx = \left[\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)\right]$

$= \lim_{t\to\infty}\frac{1}{a^2+b^2}\left[e^{-at}(-a\cos bt+b\sin bt)+a\right]$

$= \frac{1}{a^2+b^2}\left[(0\times \text{a finite quantity})+a\right] = \frac{a}{a^2+b^2}$ (finite)

$\Rightarrow \int_0^\infty e^{-ax}\cos bx\,dx$ is conv. and its value is $\frac{a}{a^2+b^2}$.

⑤ $\int_1^2 \frac{x}{\sqrt{x-1}}dx$ [Second Kind]. 1 is only P.T.D

$\int_1^2 \frac{x}{\sqrt{x-1}}dx = \lim_{\epsilon\to 0^+}\int_{1+\epsilon}^2 \frac{(x-1)+1}{\sqrt{x-1}}dx$

$$= \lim_{\epsilon\to 0^+}\int_{1+\epsilon}^{2}\left[\sqrt{x-1}+\frac{1}{\sqrt{x-1}}\right]dx$$

$$= \lim_{\epsilon\to 0^+}\left[\frac{2}{3}(x-1)^{3/2}+2\sqrt{x-1}\right]_{1+\epsilon}^{2}$$

$$= \lim_{\epsilon\to 0^+}\left[\frac{2}{3}+2-\frac{2}{3}\epsilon^{3/2}-2\sqrt{\epsilon}\right]$$

$$= \frac{8}{3} \text{ (finite).}$$

$\Rightarrow \int_{1}^{2}\frac{x}{\sqrt{x-1}}\,dx$ is convergent and its value is $8/3$.

⑥ $\int_{0}^{\pi}\frac{dx}{1+\cos x}$ [Second Kind] π is only P.I.o.D.

$$\int_{0}^{\pi}\frac{dx}{1+\cos x} = \lim_{\epsilon\to 0^+}\int_{0}^{\pi-\epsilon}\frac{dx}{2\cos^2 x/2} = \lim_{\epsilon\to 0^+}\int_{0}^{\pi-\epsilon}\frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$= \lim_{\epsilon\to 0^+}\left[\tan\frac{x}{2}\right]_{0}^{\pi-\epsilon}$$

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$$= \lim_{\epsilon\to 0^+}\left[\tan\left(\frac{\pi}{2}-\frac{\epsilon}{2}\right)\right] = \infty$$

$\Rightarrow \int_{0}^{\pi}\frac{dx}{1+\cos x}$ is divergent.

Some Tests for convergence of Integral :-

Comparision Test :- Let $I_1 = \int_a^b f(x)\,dx$, $I_2 = \int_a^b g(x)\,dx$ be two improper integrals, where

(i) 'f' and 'g' has only P.I.D at $x = a$.
(ii) 'f' and 'g' are Non-Negative i.e $f(x)$ & $g(x) \geq 0$ $\forall\ x \in [a,b]$
(iii) $f(x) \leq g(x)$ $\forall\ x \in [a,b]$.

then $\int_a^b g(x)\,dx$ is convergent $\Rightarrow \int_a^b f(x)\,dx$ is convergent

and $\int_a^b f(x)\,dx$ is divergent $\Rightarrow \int_a^b g(x)\,dx$ is divergent.

Note :- Comparision Test is same for First kind improper integrals.

Eg :- 1. $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$.

$$\frac{\sin x}{x^p} = \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x} \leq \frac{1}{x^{p-1}}$$

Let $f(x) = \frac{\sin x}{x^p}$ and $g(x) = \frac{1}{x^{p-1}}$

both $f(x)$ and $g(x)$ has P.I.D at 0.
and $f(x) \leq g(x)$ and $f(x) \geq 0$ and $g(x) \geq 0$ in $[0, \pi/2]$

So by comparision test.

Here $\int_0^{\pi/2} \frac{1}{x^{p-1}} dx$ is convergent if $p-1 < 1 \Rightarrow$ if $p<2$

$\Rightarrow \int_0^{\pi/2} \frac{\sin x}{x^p} dx$ is convergent if $\underline{\underline{p<2}}$ and divergent if $p \geq 2$.

Combarision Limit form test :-

Let $I_1 = \int_a^b f(x)dx$ and $I_2 = \int_a^b g(x)dx$ are two improper integrals, f and g defined on $[a,b]$. and $x=a$ is the only P.I.D for both f and g are Non - Negative if

$\lim_{x \to a} \frac{f(x)}{g(x)} = l$ (Non-zero and finite] , then both integrals converges or diverge together.

Case-II $I_1 = \int_a^b f(x)dx$ & $I_2 = \int_a^b g(x)dx$.

'f' and 'g' defined on $[a,\infty)$; $f(x), g(x) > 0$ and $f(x)$, $g(x)$ has NO P.I.D in $[a,\infty)$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$ [Non zero & finite] . then both behave alike i.e converge or diverge together.

Eg:- $\int_0^1 \frac{\sin^m x}{x^n}$; $m, n > 0$

$\int_0^1 \frac{\sin^m x}{x^n} = \int_0^1 \frac{\sin^m x}{x^m} \cdot \frac{1}{x^{n-m}}$

Here $f(x) = \frac{\sin^m x}{x^m} \cdot \frac{1}{x^{n-m}}$ & $g(x) = \frac{1}{x^{n-m}}$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin^m x}{x^m} \cdot \frac{1}{x^{n-m}} \cdot x^{n-m} = 1 \neq 0$ and finite

So both behave alike

$\int_0^1 \frac{1}{x^{n-m}}$ is conv if $n-m < 1 \Rightarrow \int_0^1 \frac{\sin^m x}{x^n}\,dx$ is conv. if

$n < 1+m$ and div. if $n \geq 1+m$.

Eg(2) $\int_2^3 \frac{dx}{(x-2)^{1/4}\cdot(3-x)^2}$

Here 2 and 3 are the only P.T.D on $[2,3]$.

$$\int_2^3 f(x)\,dx = \int_2^a f(x)\,dx + \int_a^3 f(x)\,dx \qquad a \in (2,3) \quad -(1)$$

To test the convergence of $\int_2^a f(x)\,dx$ at $x=2$

we take $g(x) = \frac{1}{(x-2)^{1/4}}$

$\lim_{x\to 2^+} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{1}{(3-x)^2} = 1$ which is Non-zero and finite.

By comparison test limit test both converge or diverge together So Here $\int_2^a \frac{1}{(x-2)^{1/4}}$ is conv.

$\left(n = \frac{1}{4} < 1\right)$ So $\int_2^a f(x)\,dx$ is convergent.

Now, to test the convergence of $\int_a^3 f(x)dx$ at $x=3$

take $g(x) = \frac{1}{(3-x)^2}$

$\lim_{x\to 3} \frac{f(x)}{g(x)} = \lim_{x\to 3} \frac{1}{(x-2)^{1/4}} = 1 \neq 0$ and finite so

both $\int_a^3 f(x)dx$ and $\int_a^3 g(x)dx$ behave alike. So

Here $\int_a^3 \frac{1}{(3-x)^2}$ is divergent so $\int_a^3 f(x)dx$ is

Hence from (i) $\int_2^3 f(x)dx$ is divergent.

Eg (3) $\int_0^2 \frac{\log x}{\sqrt{2-x}}\, dx$.

$\underline{\int}$

Here 0 and 2 are P.I.D of f in $[0,2]$.

Then $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$.

To test the convergence of $\int_0^1 f(x)dx$ at $x=0$

Since f(x) is Negative on $[0,1]$, we consider $-f(x)$

take $g(x) = \frac{1}{x^n}$

$\lim_{x\to 0} \frac{-f(x)}{g(x)} = \lim_{x\to 0} \frac{-x^n \log x}{\sqrt{2-x}} = 0$ if $n>0$.

$\because \lim_{x\to 0} x^n \log x = 0$ if $n>0$

taking n between 0 and 1, then integral $\int_0^1 g(x)dx$ is convergent $\Rightarrow \int_0^1 f(x)dx$ conv $\therefore \int f(x)dx$ conv.

To test the convergence of $\int_1^2 f(x)\,dx$ at $x=2$.

take $g(x) = \frac{1}{\sqrt{2-x}}$

$\lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \log x = \log 2 \neq 0$ & fin

Here $\int_1^2 g(x)\,dx$ is conv $\Rightarrow \int_1^2 f(x)\,dx$ is conv.

from (i) $\int_1^2 f(x)\,dx$ is conv. $\Rightarrow \int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$ is conv.

Note :- $\int_0^1 \frac{1}{x^p}\,dx$ is convergent if $p<1$ and divergent if $p\geq 1$.

Note :- $\int_a^\infty \frac{1}{x^p}$ $(a>0)$ is convergent if $p>1$ and divergent if $p\leq 1$.

Note :- $\int_a^b \frac{1}{(x-a)^p}\,dx$ is convergent if $p<1$ and divergent if $p\geq 1$

Note :- $\int_a^b \frac{1}{(b-x)^p}\,dx$ is convergent if $p<1$ and divergent if $p\geq 1$.

Dirichlet Test :- Let $I = \int_a^b f(x)\cdot g(x)\,dx$ if

① $\left|\int_a^t g(x)\,dx\right| \leq M \quad \forall\ t\in[a,\infty)$

② $f(x)$ is Monotonic and $\lim_{x\to\infty} f(x) = 0$ then

$\int_a^\infty f(x)\cdot g(x)\,dx$ is convergent

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Eg (1). $\int_0^\infty \frac{\sin^3 x}{x^5}\,dx$.

$= \int_0^\infty \frac{\sin^3 x}{x^3} \cdot \frac{1}{x^2}\,dx$.

Here $g(x) = \frac{\sin^3 x}{x^3}$

So $\left|\int_0^t \frac{\sin^3 x}{x^3}\right| \le M \ \forall\ t \in [0,\infty)$.

and $f(x)$ is Monotonic $\downarrow$ and $\lim_{x\to\infty} g(x) = 0$ where $g(x) = \frac{1}{x^2}$. So by Dirichlet test $\int_0^\infty \frac{\sin^3 x}{x^5}$ is conv.

② $\int_0^\infty \frac{\cos x}{x}\,dx$ Here $g(x) = \cos x$ & $f(x) = \frac{1}{x}$.

So $\int_0^\infty \frac{\cos x}{x}\,dx$ is conv. By Dirichlet test.

③ $\int_2^\infty \frac{\sin x}{\log x}\,dx$ is conv. by taking $g(x) = \sin x$ and $f(x) = \frac{1}{\log x}$

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Theorem Let 'f' and 'g' are two positive Functions on $[a,b]$, 'a' is the only P.T.D and if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ & $\int_a^b g(x)\,dx$ is convergent then $\int_a^b f(x)\,dx$ is convergent. And if $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)\,dx$ is divergent then $\int_a^b f(x)\,dx$ is divergent.

Theorem :- if 'a' is the only P.T.D of 'f' on $[a,b]$ and $\lim_{x\to a}(x-a)^{\mu} f(x)$ exists finitely and Non-zero, then $\int_a^b f(x)\,dx$ converges iff $\mu<1$

Theorem :- if 'b' is the only P.T.D of 'f' on $[a,b]$ and $\lim_{x\to b}(b-x)^{\mu}.f(x)$ exists finitely and Non-zero then $\int_a^b f(x)\,dx$ converges iff $\mu<1$.

Eg :- $\int_0^{\infty} \frac{\sin x}{x^{3/2}}\,dx$

$$\int_0^{\infty}\frac{\sin x}{x^{3/2}}\,dx = \int_0^{1}\frac{\sin x}{x^{3/2}}\,dx + \int_1^{\infty}\frac{\sin x}{x^{3/2}}\,dx.$$

So Applying Dirichlet test.

$$\left|\int_0^t \sin x\,dx\right| = \left|(-\cos x)_0^t\right| = \left|-\cos t+\cos 0\right|$$

$$\le |\cos t|+1$$

$$\le 2$$

and $g(x) = \frac{1}{x^3}$ is $\downarrow$ and $\lim_{x\to\infty}\frac{1}{x^{3/2}} = 0$ so both conditions of test hold good. Hence $\int_0^{\infty}\frac{\sin x}{x^{3/2}}\,dx$ is convergent.

loq 1.

Applying comparision limit form test in $\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$

and Dirichlet in $\int_1^\infty \frac{\sin x}{x^{3/2}}\,dx$. is Convergent. so both are conv $\Rightarrow$ $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx$ is convergent.

Absolute Convergence :- The improper integral $\int_a^\infty f(x)\,dx$ is said to be Absolute Convergent if $\int_a^\infty |f(x)|\,dx$ is cgt.

Thm:- Every Absolute convergent integral is convergent.

~~ex~~ 'f' is defined on $[1,\infty)$ s.t

$$f(x) = \begin{cases} 1/x^2 ; & x \in \mathbb{Q} \\ -1/x^2 ; & x \in \mathbb{Q}^c \end{cases}$$

then $\int_1^\infty f(x)\,dx$ is conv.

Soln:- $\int_1^\infty |f(x)|\,dx = \int_1^\infty \frac{1}{x^2}\,dx$

$\Rightarrow \int_1^\infty f(x)\,dx$ is absolute convergent.

$\Rightarrow \int_1^\infty f(x)\,dx$ is cgt.

Note:- Convergence of Integral $\not\Rightarrow$ Absolute convergent.

Eg:- $\int_1^{\infty} \frac{\sin x}{x^p}$ is convergent but

$$\int_1^{\infty}\left|\frac{\sin x}{x^p}\right| \leq \int_1^{\infty}\frac{1}{x^p}\,dx$$

$$\leq \underset{\substack{\downarrow \\ \text{conv}}}{\int_1^{1}\frac{1}{x^p}\,dx} + \underset{\substack{\downarrow \\ \text{div.}}}{\int_1^{\infty}\frac{1}{x^p}\,dx}$$

$\Rightarrow \int_1^{\infty}\left|\frac{\sin x}{x^p}\right|$ is Not cgt if $p \leq 1$.

Hence $\int_1^{\infty}\frac{\sin x}{x^p}\,dx$ converges Absolutely if $p > 1$

for $p = 1$ it is Not Absolutely convergent.

Q4 $\int_0^{\infty}(\sin x^2)\,dx$

Put $x^2 = t \Rightarrow x = \sqrt{t}$

$2x\,dx = dt \Rightarrow dx = \frac{dt}{2\sqrt{t}}$

Now $\int_0^{\infty}(\sin x^2)\,dx = \left|(-\cos x)_0^t\right| \leq 2$

$\Rightarrow \left|\int_0^t \sin x\,dx\right|$ is Bounded

and $g(t) = \frac{1}{2\sqrt{t}}$ is monotonically decreasing

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and $\lim_{t\to\infty} \frac{1}{2\sqrt{t}} = 0$ by Dirichlet's test

$\int_0 \sin x^2 \, dx$ is cgt.

Q4 Find the conditions on m & n for which $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent.

Sol^n - Let $I = \int_0^1 x^{m-1}(1-x)^{n-1}$, Now.

I can be written as

$$I = \int_0^1 \frac{1}{x^{1-m}\,(1-x)^{1-n}}$$

If we apply limit form test by taking $g(x) = \frac{1}{x^{1-m}}$ and $f(x) = I$ then we will confuse about the point where $\lim_{x\to a} \frac{f(x)}{g(x)}$ will be checked.

Because I has two point of discountinuity 0 and 1 so we need to write I as.

$$I = \int_0^{1/2} x^{m-1}(1-x)^{n-1} + \int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$$

or

$$I = \underbrace{\int_0^{1/2} \frac{1}{x^{-m+1}(1-x)^{1-n}}}_{(I_1)} + \underbrace{\int_{1/2}^1 \frac{1}{x^{1-m}(1-x)^{1-n}}}_{(I_2)}$$

Here I_1 has only P.I.P at $x=0$ & I_2 has PIP at $x=1$.

Now for I_1 :- let $f(x) = \frac{1}{x^{1-m}(1-x)^{1-n}}$

$g(x) = \frac{1}{x^{1-m}}$

Now $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{(1-x)^{1-n}} = 1 \neq 0$

Hence $\int_0^{1/2} f(x)\,dx$ & $\int_0^{1/2} g(x)$ behaves alike, Here

$\int_0^{1/2} g(x) = \int_0^{1/2} \frac{1}{x^{1-m}}$ is cgt iff $1-m < 1 \Rightarrow m > 0$

Hence $\int_0^{1/2} f(x)dx = \int_0^{1/2} I_1 dx = \int_0^{1/2} \frac{1}{x^{1-m}(1-x)^{1-n}}$ is conv. if $m > 0$

Similarily I_2 is conv. for $n > 0$

Hence $I = I_1 + I_2$ is conv. if $m > 0$ & $n > 0$

i'e $I = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ is conv. for $m, n > 0$.

Qu Discuss the convergence for integral $\int_0^1 e^{-mx} x^n dx$.

Sol:- Let $I = \int_0^1 e^{-mx} \cdot x^n dx$

if $n > 0$ then I is Proper Integral $\forall$ $m \in R$

but if $n<0$ then, we need to check.

Let $f(x) = e^{-mx}\cdot x^n$

$g(x) = x^n = \frac{1}{x^{-n}}$,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} e^{-mx} = 1 \neq 0$$

{If $n<0$ then I has only P.T.D at $x=0$, that's why we are taking lim at $x=0$}

$\Rightarrow$ both $\int_0^1 g(x)$ & $\int_0^1 f(x)dx$ behaves alike and

$\int_0^1 \frac{1}{x^{-n}}$ is conv if $-n<1 \Rightarrow n>-1$

Hence $I = \int_0^1 e^{-mx}\cdot x^n dx$ is conv. for every $n>-1$ and $\forall\, m\in\mathbb{R}$. & div if $n \leq -1$.

Some Important Improper Integrals

1) $I = \int_0^\infty \frac{x^p}{1+x^q\cos^2 x} dx$ then.

1) Convergent if $\frac{q}{2} - p > 1$

② Divergent if $\frac{q}{2} - p \leq 1$.

Eg: $I = \int_0^\infty \frac{1}{x^3\cos^2 x+1} dx$

$p=0$, $q=3$, then $\frac{q}{2} - p = \frac{3}{2} - 0 = \frac{3}{2} > 0$. then

I is cgt.

Eg:- $I = \int \frac{x^2}{1+x^5\cos^2 x}\, dx$

$p=2$, $q=5$ then $\frac{q}{2} - p = \frac{5}{2} - 2 = \frac{1}{2} < 1$ then divergent.

② $\int_0^1 x^{\alpha-1} e^{-x}\, dx$ then

1> integral is convergent if $\alpha > 0$

2> Integral is divergent if $\alpha \leq 0$

Eg:- $\int_0^1 x^2 e^{-x}\, dx$ is conv.

$I = \int_0^1 x^2 e^{-x}\, dx = \int_0^1 x^{3-1} e^{-x}\, dx$, Here $\alpha = 3 > 0$

Eg:- $\int_0^1 \frac{e^{-x}}{x}\, dx$.

$I = \int_0^1 \frac{e^{-x}}{x}\, dx = \int_0^1 x^{-1} e^{-x}\, dx$ Here $\alpha - 1 = -1$
$\Rightarrow \alpha = 0$

Hence I is divergent.

③ $I = \int_0^\infty \frac{x^{\alpha-1}}{1-x}\, dx$, then I is

1> Convergent if $0 < \alpha < 1$ 2> divergent if $\alpha \geq 1$.

FUNCTIONS OF BOUNDED VARIATION :-

The concept of Bounded variation is closely Related or associated to that of Monotonic function.

Let 'f' be a function defined on $[a,b]$. And let A set $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a Partition of $[a,b]$, then consider the Sum

$$\sum_{r=1}^{n} |f(x_r) - f(x_{r-1})| = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_{r-1}) - f(x_{r-2})| + \cdots + |f(x_n) - f(x_{n-1})|$$

and this sum is denoted by $V(f,P)$ i.e

$$V(f,P) = \sum_{r=1}^{n} |f(x_r) - f(x_{r-1})|$$

then $V(f,P)$ is called variation of 'f' over the Partition P.

Result :- $V(f,P^*) \geq V(f,P)$ where P^* is Refinement of P.

Let smaller Partition P with 2 points set $P = \{a,b\}$ then variation of f, $V(f,P) = |f(x_1) - f(x_0)|$

$$= |f(b) - f(a)| \quad \text{—(1)}$$

Now, define Refinement with at least one additional point.

i.e $P^* = \{a = x_0, x_1, x_2 = b\}$.

$$\begin{aligned} V(f,P) &= |f(x_2) - f(x_0)| \\ &= |f(x_2) - f(x_1) + f(x_1) - f(x_0)| \\ &\le |f(x_2) - f(x_1)| + |f(x_1) - f(x_0)| \\ &\le V(f,P^*) \end{aligned}$$

Hence $V(f,P^*) \ge V(f,P)$

Result:- $V(f,P) \ge 0$

Result:- If f is a constant function then $V(f,P) = 0$

Result:- If c is any point s.t. $x \le c \le y$

$$|f(y) - f(x)| \le |f(y) - f(c)| + |f(c) - f(x)|$$

Conclusion:- The sum $\sum |f(x_i) - f(x_{i-1})|$ cannot decrease by the introduction of additional points to a partition of interval.

Qu $f : [0,1] \to \mathbb{R}$ s.t. $f(x) = x$, then

let $P = \{x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 3/3 = 1\}$

$$V(f,P) = \sum_{i=1}^{3} |f(x_i) - f(x_{i-1})| = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + |f(x_3) - f(x_2)|$$

$$V(f,P) = \sum_{i=1}^{3} |f(x_i) - f(x_{i-1})|$$

$$= \left|\frac{1}{3} - 0\right| + \left|\frac{2}{3} - \frac{1}{3}\right| + \left|1 - \frac{2}{3}\right|$$

$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

$V(f,P) = 1$

This is the variation of $f(x) = x$ over the Partition $P = \{0, 1/3, 2/3, 1\}$ of $[0,1]$.

Now if $P^* = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

then $V(f,P^*) = \left|\frac{1}{4} - 0\right| + \left|\frac{2}{4} - \frac{1}{4}\right| + \left|\frac{3}{4} - \frac{2}{4}\right| + \left|1 - \frac{3}{4}\right|$

$V(f,P^*) = 1$.

Total variation :- Let $V(f,P) = \sum_{r=1}^{n} |f(x_r) - f(x_{r-1})|$ be variation of (f) over Partition P of $[a,b]$

As we know, there are infinite Partitions of $[a,b]$ and collect all the variation of (f) over infinite partitions of $[a,b]$. So the set of all these variations or sums will be infinite.

If the set of sums is Bounded above, then (f) is said to be of Bounded variation. And the supremum of the set of sums is called the total variation of (f) on $[a,b]$. And denoted by $V(f)$ or $V(f,[a,b])$.

i.e. $\sup_P [V(f,P)]$ [if exists finitely] $= V[f,[a,b]]$ = Total Variation

In all,

The function 'f' is defined on [a,b] is said to be bounded variation if $V[f,[a,b]]$ is finite i.e $\sup_P V(f,P) < +\infty$.

Result:- If 'f' is bdd and monotonic on [a,b] then 'f' is Bounded variation.

Note:- Continuity is Neither sufficient Nor Necessary condition for a function 'f' to be of Bounded variation.

Eg:-

$$f(x) = \begin{cases} x \sin \frac{\pi}{x} , & 0 < x \leq 1 \\ 0 ; & x = 0 \end{cases}$$

Here 'f' is continuous on [0,1].

Let $P = \left\{ 0, \frac{2}{2n+1}, \frac{2}{2n-1}, \ldots, \frac{2}{5}, \frac{2}{3}, 1 \right\}$ be a Partion of [0,1].

Now.

$$\sum |f(x_r) - f(x_{r-1})| = |f(1) - f(2/3)| + |f(2/3) - f(2/5)|$$

$$+ \ldots \ldots \left| f\left(\frac{2}{2n-1}\right) - f\left(\frac{2}{2n+1}\right) \right| +$$

$$\ldots + \left| f\left(\frac{2}{2n+1}\right) - f(0) \right|$$

$$= \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{5}\right) + \left(\frac{2}{5} + \frac{2}{7}\right) + \ldots +$$

$$-+\left(\frac{2}{2n+1}+\frac{2}{2n+1}\right)+\frac{2}{2n+1}$$

$$=4\left[\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}\right.$$

Here Infinite series $\frac{1}{3}+\frac{1}{5}+\cdots$ is divergent. Because it's SOPS $\langle S_n\rangle=\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}$ is

Not Cauchy so Not cgt ; $\langle S_n\rangle$ is divergent.

So $\sum|f(x_r)-f(x_{r-1})|$ is Not finite, Hence 'f' is Not of Bounded variation, even if f is cts, uniformly continuous, Reimann integrable on $[a,b]$.

Eg ② $f(x)=[x]$ is Montonic on $[0,2]$ So 'f' is bdd variation, but 'f' is Not continuous.

Hence by e.g. ① & (2), we can say that continuity is Neither sufficient Nor Necessary condition for a function 'f' to be bounded variation.

Result if the Derivative f' of function f is Bounded on $[a,b]$ then the fun. is of Bounded variation.

Note :- Boundedness of f' is sufficient condition Not Necessary.

Eg :- $x^{1/3}$, $\because x^{1/3}$ is Monotonic.

$\Rightarrow x^{1/3}$ is of B.V on $[0,1]$.

but $f'(x) = \frac{1}{3x^{2/3}} \to \infty$ at $x \to 0$

Result :- ① A function 'f' is of Bounded Variation is necessarily Bounded.

Result :- ② $v[f,P] \geq |f(b)-f(a)|$

or $v[f,[a,b]] \geq |f(b)-f(a)| \ \forall P$

Result → ③ $v[f,[a,b]] = v[f,[a,c]] + v[f,[c,b]]$.

Result → ④ A function 'f' is of Bounded variation then 'f' is Reimann Integrable, Not conversely.

Properties of Variation Function :-

(i) $V(a) = 0$

(ii) $V(b) =$ total Variation

(iii) $V_f(x)$ is monotonically increasing

(iv) If f is B.V on $[a,b]$ then $V_f(x)+f(x)$ and $V_f(x)-f(x)$ is a Monotonic increasing function.

v) $V_f(x)$ is cts iff f is cts.

Result ! — if f is R.I then its Indefinite integral

$$F(x) = \int_a^x f(t)\,dt$$ is of Bounded variation.

Result :- if $V[f, [a,b]] = \int_a^b |g(x)|\,dx$

then $v[f, [a,b]] = \int_a^b |f'(x)|\,dx$

Let g is cts on $[a,b]$ & $f(x) = \int^x g(x)\,dx$ if

f is diff on $[a,b]$ and $f'(x) = g(x)$

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Let $P = \{a = x_0 = x_1, - - x_n = b\}$ be a Partition of $[a,b]$

Now $\dfrac{f(x_r) - f(x_{r-1})}{\Delta x_r} = f'(t_r) = g(t_r)$

$\because$ g is cts so $g \in R[a,b]$

$f(x_r) - f(x_{r-1}) = g(t_r)\cdot\Delta x_r$

$|f(x_r) - f(x_{r-1})| = |g(t_r)|\cdot\Delta x_r$

$\|P\| \to 0$

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$v[f, [a,b]] = \int_a^b |f(x)|\,dx$

$$\boxed{v[f, [a,b]] = \int_a^b |f'(x)|\,dx}$$

i.e. if $f \in R[a,b]$ then we can find total variation of f in the given interval by above formula.

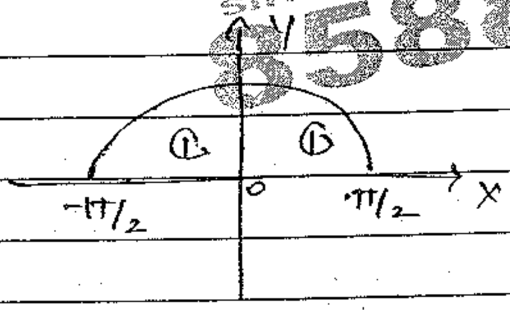
Eg: $f(x) = \cos x$; $x \in [-\pi/2, \pi/2]$ find total variation

Sol^n :- ∵ f is cts on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $f \in R\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

the total variation $= \int_{-\pi/2}^{\pi/2} |f'(x)|\, dx$,

$$= 2\int_0^{\pi/2} |-\sin x|\, dx = 2\int_0^{\pi/2} \sin x\, dx$$

$$= 2\left[-\cos x\right]_0^{\pi/2} = 2,$$

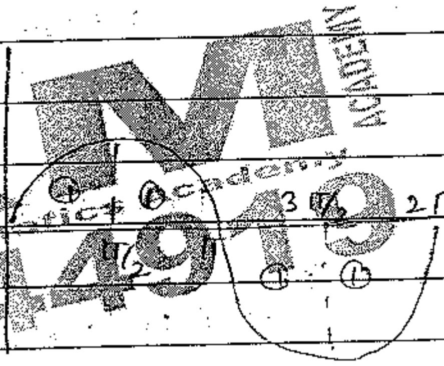


i.e 1 variation from $-\frac{\pi}{2}$ to 0 and 1 from 0 to $\frac{\pi}{2}$ so

total variation = 2 .

Q. $f(x) = \sin x$; $x \in [0, 2\pi]$

Sin x is Monotonic increasing in $[0, \pi/2]$ and $[3\pi/2, 2\pi]$ and decreasing in $[\pi/2, \pi]$ and (π to $3\pi/2$) Hence $f'(x) = \cos x$



So total variation

$$V[f; [0, 2\pi]] = V[f_1[0, \pi/2]] + V[f_1[\pi/2, \pi]] + V\left[f_2\left[\pi, \frac{3\pi}{2}\right]\right] + V\left[f_2\left[\frac{3\pi}{2}, 2\pi\right]\right]$$

$$= \int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x| + \int_{\pi}^{3\pi/2} |\cos x| + \int_{3\pi/2}^{2\pi} |\cos x|$$

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$= 1+1+1+1 = 4$ Ans.

Positive variation and Negative variation function.

Let f be a function of Bounded variation on $[a,b]$ Now define two function $p(x)$ and $q(x)$ s.t

$f(x) - f(a) = p(x) - q(x)$ and

$v(x) = p(x) + q(x)$; then $p(x)$ and $q(x)$ are called positive and Negative variation function Respectively.

Properties 1) $p(a) - q(a) = 0$

2) Function $p(x)$ and $q(x)$ are Non Decreasing Monotonically function.

Result :- if function is Lipschitz function then 'f' is of Bounded variation.

Result :- Let $f : [0,1] \to \mathbb{R}$, $\alpha, \beta > 0$ s.t

$$f(x) = \begin{cases} x^{\alpha} \sin \frac{1}{x^{\beta}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{or} \quad \begin{cases} x^{\alpha} \cos \frac{1}{x^{\beta}}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$

then 'f' is of Bounded variation if $\alpha > \beta$ and 'f' is Not of Bounded variation if $\alpha \leq \beta$.

Result :- if $f(x) = ax^2 + bx + c$, $x \in [\alpha, \beta]$ then find $f'(x) = 0 \Rightarrow x = \frac{-b}{2a}$ Now

$$V[f, [\alpha, \beta]] = \left| f\left(-\tfrac{b}{2a}\right) - f(\alpha) \right| + \left| f(\beta) - f\left(-\tfrac{b}{2a}\right) \right|$$

Eg:- Find total variation for $f(x) = x^2 - 5x + 6$ in $[0,4]$.

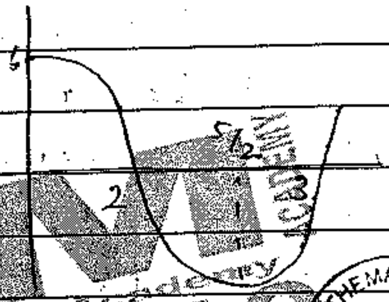
Sol^n: $f(x) = (x-2)(x-3)$

$f(0) = 6$

$f'(x) = 0$

$\Rightarrow 2x - 5 = 0$

$x = 5/2$.



$$v.[f,[0,4]] = \left|f(5/2) - f(0)\right| + \left|f(4) - f(5/2)\right|$$

$$= \left|-\frac{1}{4} - 6\right| + \left|2 + \frac{1}{4}\right| = \frac{34}{4}.$$

$$= \frac{17}{2} = 8.5$$ Ans

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① Let $f : [0,1] \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 1 & ; \quad 0 \le x \le 0.5 \\ 2 & ; \quad 0.5 \le x \le 0\text{[illegible]} \\ 3 & \quad 0.7 < x \le 1 \end{cases}$$

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then.

(a) f is Not R.I

(b) f is R.I and $\int_0^1 f(x)\,dx = 1.8$

(c) f is R.I and $\int_0^1 f(x)\,dx = 2.1$

(d) None of these.

Que-2 f is Real valued function on $[0,1]$; consider the statements.

1 f is R.I iff f is cts.

2) f is R integrable iff f has finitely many discts.

then

a) 1 is true & 2 is false

b) 2 is true & 1 is false

[✓ c)] Both 1 & 2 are false

d) both 1 and 2 are correct

Q-3 Which of the following is / are either Monotonic or function of Bounded variation on $[-1,1]$.

a) $f(x) = x - [x]$

b) $f(x) = \begin{cases} x \sin 1/x & ; x \ne 0 \\ 0 & ; x = 0 \end{cases}$

c) $f(x) = \sinh x$

d) $f(x) = \cosh x$

4 . choose the correct statements.

a) If $f(x) > 0$ is cts function on $[a,b]$ & $\int_a^b f(x)\,dx = 0$ then $f(x) \equiv 0$ on $[a,b]$

(b) If $f(x) \geq 0$ is cts function on $[a,b]$ s.t $\int_a^b f(x)dx = 0$ then $\exists$ atleast one point $x_0 \in [a,b]$ s.t $f(x_0) \neq 0$

(c) let $f : [0,1] \to \mathbb{R}$ s.t $f(x) = \begin{cases} \sin 1/x & \forall x \in \mathbb{Q}^c \cap [0,1] \\ 0 & \forall x \in \mathbb{Q} \cap [0,1] \end{cases}$

then f is R.I on $[0,1]$.

(d) let $f : [0,1] \longrightarrow \mathbb{R}$ s.t $f(x) = \begin{cases} \sin 1/x & \forall x \in \mathbb{Q}^c \cap [0,1] \\ 0 & \forall x \in \mathbb{Q} \cap [0,1] \end{cases}$

then f is Not R.I on $[0,1]$.

(Q-15) Let $f : [a,b] \longrightarrow \mathbb{R}$, $g : [a,b] \longrightarrow \mathbb{R}$ are bdd functions and $P \in P[a,b]$ then

a) $U(P,f+g) \leq U(P,f) + U(P,g)$
b) $L(P,f+g) \geq L(P,f) + L(P,g)$
c) $L(P,-f) = -U(P,f)$
d) $m(b-a) < L(f,P) < U(f,P) < M(b-a)$

Q-16 Let $f : \mathbb{R} \to \mathbb{R}$ be a diff function s.t f' is bdd. Given a closed and bdd interval $[a,b]$, and a partition $P = \{a_0 = a < a_1 < a_2 \cdots < a_n = b\}$ of $[a,b]$. let $M(f,P)$ and $m(f,P)$ denote

Upper and lower Reimann Sum Respectively. Then

a) $\left| M(f,P) - \int_a^b f(x)\,dx \right| \leq (b-a) \sup \{ |f(x)|, x \in [a,b] \}$

b) $\left| m(f,P) - \int_a^b f(x)\,dx \right| \leq (b-a) \inf \{ |f(x)|, x \in [a,b] \}$

c) $\left| M(f,P) - \int_a^b f(x)\,dx \right| \leq (b-a)^2 \sup \{ |f'(x)|; x \in [a,b] \}$

d) $\left| m(f,P) - \int_a^b f(x)\,dx \right| \leq (b-a)^2 \operatorname{Inf} \{ |f'(x)|; x \in [a,b] \}$

Q-) $f(x) = \begin{cases} x^2 ; & x \in Q \\ -x^2 ; & x \in Q^c \end{cases}$ on $[0,1]$

$g(x) = \begin{cases} 1/n ; & x \in \frac{1}{n+1} \leq x \leq 1/n \\ 0 ; & \text{else} \end{cases}$ on

$[0,1]$, then

a) f(x) is R.I But g(x) is Not.
b) g(x) is R.I but f(x) is Not.
c) both f(x) and g(x) are R-I.
d) Neither f(x) & g(x) is R-RI.

Q-) A function of Bounded Variation.
a) is Necessarily continuous Everywhere.
b) can have atmost c'bly many points of discontinuity.
c) can have unc'ble many points of Discontinuity
d) can Never be continuous.

Q: c is a fixed point in [0,1]. A function $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = \begin{cases} c & \text{if } 0 \le x < c \\ 2c & c < x \le 1 \end{cases}$

It is given that ~~is~~ the R. Integral

$\int_0^1 f(x)\,dx$ is $\frac{7}{16}$. Then the value of c is

(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5

Q-112 The limit $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2}$ exist and is the

Reimann Integral of

a) $f(x) = x^2$, $0 \le x \le 1$ (b) $f(x) = k$, $0 \le x \le 1$

c) $f(x) = 3x$, $0 \le x \le \frac{1}{3}$ d) $f(x) = x$, $0 \le x \le 1$

Q-113 Let $f_n : [0,1] \to \mathbb{R}$ be a seqⁿ of differentiable functions. Assume that (f_n) converges uniformly on [0,1] to a function f, then

a) f is diff. and R.I

b) f is U.C and R.I

c) f is cts, Need Not be diff on (0,1) and Need Not be R.I on [0,1]

d) f Need Not be U.C on [0,1]

Q-114 Let $f : [0,1] \to \mathbb{R}$ be cts s.t $f(t) \ge 0$ ∀ t in [0,1]. define $g(x) = \int_0^x f(t)\,dx$ then

a) g is Monotonic and bdd.
b) g is " and Not bdd.
c) g is bdd but Not monotone.
d) g is Neither Monotone nor Bounded.

Q-115 Let f be a cts function on $[0,1]$ with $f(0)=1$. Let $G(a)=\frac{1}{a}\int_0^a f(x)dx$ then.

(a) $\lim_{a\to 0} G(a)=\frac{1}{2}$ (b) $\lim_{a\to 0} G(a)=1$

(c) $\lim_{a\to 0} G(a)=0$ (d) $\lim_{a\to 0} G(a)$ doesn't Exist.

Q-116 If for $x\in\mathbb{R}$, $\phi(x)$ denotes the integer closed to x. (if there are such two integer take the larger one) then $\int_{10}^{12}\phi(x)dx$ equals

ⓐ 22 ⓑ 11 ⓒ 20 ⓓ 12

Q-117 Let $f:\mathbb{R}\to\mathbb{R}$ be a cts fun. with Period $p>0$ then $g(x)=\int_x^{x+p}f(t)dt$ is a.

a) constant function.
b) cts function.
c) cts fun. but Not Diff.
d) Neither cts Nor Diff.

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Sequence and Series of Functions :-

Let f_n be a Real valued function defined on Interval (or a subset of $\mathbb{R}$) and for each $n \in \mathbb{N}$ then the expression $\langle f_1, f_2, f_3, f_4, \text{---} \rangle$ is called a sequence of Real valued function on I. It is denoted by $\{ f_n : I \rightarrow \mathbb{R}, n \in \mathbb{N} \}$ or $\langle f_n(x) \rangle$.

(or)

Let F be a set of functions defined on $[a,b]$ or subset S of $\mathbb{R}$.

i.e $F = \{ f ; f : [a,b] \rightarrow \mathbb{R} \text{ or } f : S \rightarrow \mathbb{R} \}$

Now define

$\phi : \mathbb{N} \longrightarrow F$

$\phi(n) = f_n(x)$ Here $\phi(n) \in F$

then $\langle \phi(n) \rangle = \langle f_n(x) \rangle$ is called sequence of functions.

Eg:- $f_n(x) = x^n$ in $[0,1]$ then

$\langle f_1, f_2, f_3, \text{----} \rangle = \langle x, x^2, x^3, \text{----} \rangle$

Here $x, x^2, x^3, \text{---}$ are all functions so $\langle f_n(x) \rangle = \langle x^n \rangle$ is called sequence of Real valued function.

Note: Sequence $\langle f_n(x) \rangle$, $x \in S$ can be taken as family of sequence of Real Numbers. And different members of this family can be obtained by choosing x from S whereas different term of member sequence are obtained by varying n.

E.g.: $F = \{ f \,;\, f : S = \{1,2,3\} \longrightarrow \mathbb{R} \}$

$\phi : \mathbb{N} \longrightarrow F$ set

$\phi(n) = f_n(x) = \sin \frac{x}{n}$

$\langle f_n(x) \rangle = \langle \sin \frac{x}{n} \rangle, \; x \in S = \{1,2,3\}$

So member of this sequence $\langle f_n(x) \rangle$ by choosing x.

$\langle f_n(x) \rangle = \langle \sin \frac{1}{n} \rangle$ at $x = 1$

$\langle f_n(x) \rangle = \langle \sin \frac{2}{n} \rangle$ at $x = 2$

$\langle f_n(x) \rangle = \langle \sin \frac{3}{n} \rangle$ at $x = 3$.

Now different term of sequence (member sequence) can be obtained by choosing x

$\langle \sin \frac{1}{n} \rangle = \langle \sin 1, \sin \frac{1}{2}, \sin \frac{1}{3}, \ldots \rangle = \langle a_n \rangle$

$\langle \sin \frac{2}{n} \rangle = \langle \sin 2, \sin 1, \sin \frac{2}{3}, \ldots \rangle = \langle b_n \rangle$

$\langle \sin \frac{3}{n} \rangle = \langle \sin 3, \sin \frac{3}{2}, \sin 1, \sin \frac{3}{4}, \ldots \rangle = \langle c_n \rangle$

So sequence of functions are only family of sequence of Real Numbers.

Note:- $\langle f_n(x) \rangle$, $x \in S$, which is family of seq^n of Real Numbers, then if $\alpha \in S$ then

$\langle a_n \rangle = \langle f_n(x) \rangle$ is called point sequence.

Pointwise limit of sequence of functions $\langle f_n(x) \rangle$:-

Let $\langle f_n(x) \rangle$, $x \in S$ be the sequence of functions if for each $\alpha \in S$, Real sequence $\langle a_n \rangle = \langle f_n(\alpha) \rangle$ is convergent. Then we say pointwise limit (PWL) exists for $\langle f_n(x) \rangle$, $x \in S$.

i.e :- if sequence $\langle f_n(\alpha) \rangle \longrightarrow l_\alpha$ as $n \to \infty$ $\forall$ $\alpha \in S$, then we say $\langle f_n(x) \rangle$ is Pointwise convergent. And l_α is called pointwise limit for $\alpha \in S$.

i.e. we define a Rule [function].

$f : S \longrightarrow \mathbb{R}$ s.t

$f(\alpha) = \lim a_n$ where $\langle a_n \rangle = \langle f_n(\alpha) \rangle$

$= l_\alpha$.

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then 'f' is defined $\forall$ $x \in S$ (if posible) then 'f' is called pointwise limit and $\langle f_n(x) \rangle$ called pointwise convergent. And written as $\lim_{n \to \infty} f_n(x) = f(x)$ $\forall$ $x \in S$.

Pointwise convergent :- let $D \subseteq \mathbb{R}$, $f_n : D \longrightarrow \mathbb{R}$ is seqn of function i.e $\langle f_n(x) \rangle$. seqn of function $\langle f_n(x) \rangle$ is said to be pointwise convergent if for each $x \in D$ seqn of f_n is converges.

$\langle f_n(x) \rangle = \{ f_1(x), f_2(x), \text{-----}, f_n(x), \text{---} \}$

Eg :- (i) $\langle f_n(x) \rangle = \langle x^n \rangle$ in $[0,1]$

then $\langle f_n(0)\rangle = \langle 0^n\rangle \to 0$ at $x=0$ and

$\langle f_n(1)\rangle = \langle 1^n\rangle \to 1$ at $x=1$

$\langle f_n(\frac{1}{2})\rangle = \langle \frac{1}{2^n}\rangle \to 0$ at $x=\frac{1}{2}$

$\langle f_n(\frac{1}{3})\rangle = \langle \frac{1}{3^n}\rangle \to 0$ at $x=\frac{1}{3}$

ie $\langle f_n(x)\rangle = \langle x^n\rangle \to 0$ at every $x\in(0,1)$.

So Pointwise limit is defined as

$$f(x)=\begin{cases}1, & x=1\\ 0, & x\in[0,1)\end{cases}$$

i.e $\lim\limits_{n\to\infty}\langle f_n(x)\rangle = \lim\limits_{n\to\infty}\langle x^n\rangle = \begin{cases}1; & x=1\\ 0; & x\in[0,1)\end{cases}$

② $\langle f_n(x)\rangle = \left\langle \frac{x}{1+nx}\right\rangle$; $x\in[0,\infty)$

if we choose any $\alpha\in[0,\infty)$ seq $\langle f_n(\alpha)\rangle = \left\langle\frac{\alpha}{1+n\alpha}\right\rangle$ $\to 0$ as $n\to\infty$ So

$f(x)=0 \ \forall x\in[0,\infty)$.

③ $\langle f_n(x)\rangle = \sin x + \frac{x}{n}$; x [illegible]

Let any $\alpha\in\mathbb{R}$ seq^n $\langle f_n(\alpha)\rangle = \langle a_n\rangle = \left\langle \sin\alpha + \frac{\alpha}{n}\right\rangle$

So $\left\langle \sin\alpha + \frac{\alpha}{n}\right\rangle \to \sin\alpha$ as $n\to\infty$

$f(x) = \sin x \quad \forall \quad x\in\mathbb{R}$.

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4> $\langle f_n(x)\rangle = \left\langle \frac{x}{n} \right\rangle$

$\lim_{n\to\infty} \frac{x}{n} = 0 = f(x)$. point wise limit.

$\Rightarrow f(x) = 0 \quad \forall \; x \in \mathbb{R}$.

5> $\langle f_n(x)\rangle = \left\langle \frac{nx}{1+n^2x^2} \right\rangle \; ; \; x \in \mathbb{R}$

$\langle f_n(x)\rangle = \left\langle \frac{1}{nx + 1/nx} \right\rangle \; ; \; x \in \mathbb{R}$.

let any $\alpha \in \mathbb{R}$. $\langle f_n(\alpha)\rangle = \left\langle \frac{1}{n\alpha + 1/n\alpha} \right\rangle \to 0$ as $n \to \infty$ so p.w.L $= f(x) = 0 \quad \forall \; x \in \mathbb{R}$.

6) $\langle f_n(x)\rangle = \langle e^{-nx} \rangle$ for $x \geq 0$

P.w.L $= f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-nx} = \begin{cases} 1 \; ; & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

(7) $\langle f_n(x)\rangle = \left\langle \frac{x^n}{n} \right\rangle$ on $(-\infty, \infty)$

limit function (P.w.L) $f(x) = \begin{cases} \infty & ; \; |x| > 1 \\ 0 & ; \; |x| \leq 1 \end{cases}$

⑧ $\langle f_n(x)\rangle = \langle \tan^{-1} nx \rangle \; ; \; x \geq 0$

$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1} nx \begin{cases} \tan^{-1}\infty = \pi/2 & \text{if } x > 0 \\ \tan^{-1} 0 = 0 & \text{if } x = 0 \end{cases}$

So $f(x) = \begin{cases} \pi/2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

① $f_n(x) = \begin{cases} n^2 x & ; \ 0 \le x < 1/n \\ -n^2 x + 2n & ; \ \frac{1}{n} \le x < \frac{2}{n} \\ 0 & ; \ \frac{2}{n} \le x \le 1 \end{cases}$ Here $x \in [0,1]$

$F_1(x) = \left.\begin{cases} x , & 0 \le x < 1 \\ -x+2 , & 1 \le x < 2 \\ 0 , & 2 < x < 1 \end{cases}\right\}$ Here $[0,1] \nsubseteq [1,2]$.

So $f_1(x) = x \quad \forall \ x \in [0,1]$.

$f_2(x) = \begin{cases} 4x & 0 \le x < 1/2 \\ -4x+4 & 1/2 \le x < 1 \\ 0 & 1 \le x \le 1 \end{cases}$

$f_3(x) = \begin{cases} 9x & ; \ 0 \le x < 1/3 \\ -9x+6 & ; \ 1/3 \le x < 2/3 \\ 0 & \ \ 2/3 \le x \le 1 \end{cases}$

$f_4(x) = \begin{cases} 16x , & 0 \le x < 1/4 \\ -16x+8 & ; \ 1/4 \le x < 1/2 \\ 0 & ; \ \frac{1}{2} < x < 1 \end{cases}$

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and So on.

As we approches to a large $n \in \mathbb{N}$ then both

interval become smaller $\{0 \le x < \frac{1}{n}, \frac{1}{n} \le x < \frac{2}{n}\}$ and last sub-interval become large i.e.
when we will aproach ∞ i.e $n \to \infty$, last sub interval will contains all the points of Domain and in this subinterval, value of function is zero

So P.W.L $= \lim_{n\to\infty} f_n(x) = 0$

⑩ $\langle f_n(x) \rangle = \langle n x \cdot e^{-nx^2} \rangle$, $x \ge 0$

Limit function $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{nx}{e^{nx^2}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{x}{x^2 \cdot e^{nx^2}} = 0 \quad (\text{L-Hospital})$$

Hence $f(x) = 0 \quad \forall \; x \in [0, \infty)$

Classical Meaning of Pointwise Limit :-

If 'f' is the pointwise limit of a sequence of functions $\langle f_n(x) \rangle$ defined on S, then to each $\epsilon > 0$ and to each $x \in [a,b] = S$, $\exists$ integer m (depends on ϵ and x) s.t $|f_n(x) - f(x)| < \epsilon \;\; \forall \; n \ge m \;\; \forall \; x \in S$.

i.e if PWL $f(x)$ exist then we can find $m \in \mathbb{N}$ for each given $\epsilon > 0$ or we can find m s.t after m^{th} term the difference between any member sequence and PWL can be made as small as we can do.

Note :- $\forall\ x \in S$ we have a Real sequence $\langle a_n \rangle = \langle f_n(x) \rangle$
So for different $x \in S$ and $\epsilon > 0$, m exists and it can be different for different $x \in S$ and $\epsilon > 0$ i.e. m depends on both ϵ and $x \in S$.

Algorithem to show f(x) is pointwise limit :-

Step→1 Simplify $|f_n(x) - f(x)|$ to get $|f_n(x) - f(x)| \leq \phi(x,n)$

($\phi(x,n)$ function of x and n)

Step→2. Solve $\phi(x,n) < \epsilon$ to get $n > \psi(x,\epsilon)$ Here $\psi(x,\epsilon)$ is function of x and ϵ.

Step→3 :- if $\psi(x,\epsilon)$ is defined $\forall\ x \in S$ or I and $\forall\ \epsilon > 0$ then set $m = [\psi(x,\epsilon)]$ {greatest integer}

{ i.e find supremum of $\psi(x,\epsilon)$ in I say $m = [m(\epsilon)]$ }

claim :- $n \geq m > [\psi(x,\epsilon)]$

$|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq m$

Eg:- $\langle f_n(x) \rangle = \langle x^n \rangle$; $x \in [0,1]$

Limit function $f(x) = \begin{cases} 1 & ; x = 1 \\ 0 & ; x \in [0,1) \end{cases}$

Now $|f_n(x) - f(x)| = |x^n - 0|$

$= |x^n| < \epsilon$

$= x^n < \epsilon$ $\left(\because |x|^n = x^n \ \forall x \ge 0\right)$

$\Big\{$ step I for getting $\phi(x,n)$ Here $\phi(x,n) = x^n$ $\Big\}$

For Step→2. $\because x^n < \epsilon$

$\Rightarrow n\log x < \log \epsilon$

$-n \log x > -\log \epsilon$

$n \log \frac{1}{x} > \log \frac{1}{\epsilon}$

$n > \dfrac{\log 1/\epsilon}{\log 1/x}$ $\left(\text{Here } \psi(x,\epsilon) = \dfrac{\log 1/\epsilon}{\log 1/x} \right.$

For Step→3 This $\psi(x,\epsilon)$ is defined $\forall \epsilon > 0$ & $\forall x \in [0,1)$ set $m = \left[\dfrac{\log 1/\epsilon}{\log 1/x} \right]$

Claim $\forall n \ge m$; $|f_n(x) - f(x)| < \epsilon$.

Note:- Here $\psi(x,\epsilon)$ depends on both x and ϵ. So $\langle f_n(x) \rangle = \langle x^n \rangle$ pointwise convergent and pointwise (limit function)

$f(x) = \begin{cases} 1, & x = 1 \\ 0, & \text{else} \end{cases}$

Uniform Convergent :- A sequence of function $\langle f_n(x) \rangle$ is said to be converges uniformely on an interval I to a limit function $f(x)$ if for any $\epsilon > 0$ and $\forall\, x \in I$, there exist an integer m (depend only on ϵ, Not on x) s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \ge m,\ \forall\, x \in I$$

Then we say $\langle f_n(x) \rangle$ is uniformely convergent and pointwise limit is called uniform limit.

Algorithem to show $\langle f_n(x) \rangle$ is uniform convergent:-

Step 1:- Solve $|f_n(x) - f(x)|$ to get $\phi(x, n) < \epsilon$

Step 2: Solve $\phi(x, n) < \epsilon$ to get $n > \psi(x, \epsilon)$

Step 3: find the sup of $\psi(x, \epsilon)$ on Interval I

i.e. $\sup_{x \in I} \psi(x, \epsilon) \le M(\epsilon)$ [function of ϵ only]

Step 4: set $m = \lceil M(\epsilon) \rceil$ and claim $n > m$ for which $|f_n(x) - f(x)| < \epsilon$

Important Note:- if you are sure for step 3 for getting $M(\epsilon)$ i.e. if you are doubtfull to get the sup of $\psi(x, \epsilon)$ on Interval I, then don't go through this Algorithem. Your answer may be incorrect. In all, if your are very sure about your ϵ-function $M(\epsilon)$, then go ahead otherwise use M_n-Test.

NET
Que (1) Show that $\langle x^n \rangle$ is uniformely convergent on $[0, a]$, $a < 1$ but Not on $[0, 1]$ or on $(0, 1)$.

Soln:- $\langle f_n(x) \rangle = \langle x^n \rangle$, $f(x) = \begin{cases} 1, & x = 1 \\ 0, & \text{else} \end{cases}$

$|f_n(x) - f(x)| = |x^n - 0| = x^n < \epsilon$

$\Rightarrow n \log x < \log \epsilon$

$n > \dfrac{\log 1/\epsilon}{\log 1/x}$

Here $\psi(x,\epsilon) = \dfrac{\log 1/\epsilon}{\log 1/x}$

Sup of $\psi(x,\epsilon)$ can be obtained on $[0,a]$ but Not $[0,1]$ or on $(0,1)$.

" sup of $\dfrac{\log 1/\epsilon}{\log 1/x}$ Can be obtained taking $\log 1/x$ smaller. And $\log 1/x$ is decreasing function. So it will given minimum at the end point of interval. It interval contained 1 then n can't be determined so we Need to take any other point which is less than 1 let $a < 1$. "

Here $\sup\limits_{x \in [0,a]} \psi(x,\epsilon) = \dfrac{\log 1/\epsilon}{\log 1/a}$

So $n > \left[\dfrac{\log 1/\epsilon}{\log 1/a}\right] = m$ (depend only ϵ, Not on x)

Hence $\langle x^n \rangle$ is U.C on $[0,a]$, $a<1$ and not U.C on $[0,1]$ or on $(0,1)$.

Que:- $\langle f_n(x) \rangle = \left\langle \dfrac{x}{1+nx} \right\rangle$, $x \in \mathbb{R}$

For limit function, let any $\alpha \in \mathbb{R}$ point seqn.

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$\langle f_n(x)\rangle = \left\langle \frac{x}{1+nx} \right\rangle \to 0$ as $n \to \infty$

Hence $f(x) \equiv 0 \ \forall x \in \mathbb{R}$.

Now $|f_n(x) - f(x)| = \left|\frac{x}{1+nx} - 0\right| = \left|\frac{1}{\frac{1}{x}+n}\right| \le \frac{1}{n} < \epsilon$

Here $n > \frac{1}{\epsilon}$, set $m = \left[\frac{1}{\epsilon}\right]$ depend only on ϵ

Hence $\left\langle \frac{x}{1+nx} \right\rangle$ is u.c on IR.

Qu $\langle f_n(x)\rangle = \left\langle \frac{1}{x+n} \right\rangle$, $x \in \mathbb{R}$.

For limit function let $x \in \mathbb{R}$
point seqn $\langle f_n(x)\rangle = \left\langle \frac{1}{x+n} \right\rangle \to 0$ as $n \to \infty$

Hence $f(x) \equiv 0 \ \forall x \in \mathbb{R}$

Now $|f_n(x) - f(x)| = \left|\frac{1}{x+n} - 0\right| = \left|\frac{1}{x+n}\right| < \frac{1}{n} < \epsilon$

$n > \frac{1}{\epsilon}$, set $m = \left[\frac{1}{\epsilon}\right]$ depends only on ϵ

Hence $\left\langle \frac{1}{x+n} \right\rangle$ is u.c on IR

Qu $\langle f_n(x)\rangle = \langle \tan^{-1} nx \rangle$; $x \in [0,1]$

Limit function, $f(x) = \begin{cases} 0 & ; x = 0 \\ \pi/2 & ; \text{else} \end{cases}$

$|f_n(x) - f(x)| = \left|\tan^{-1} nx - \frac{\pi}{2}\right| = |\cot^{-1} nx| < \epsilon$

$\Rightarrow \cot nx < \epsilon$

$nx > \cot \epsilon$

$n > \frac{\cot \epsilon}{x}$

$\psi(x, \epsilon) = \frac{\cot \epsilon}{x}$ will be larger when x become smaller

$\because x \in [0,1]$ so $\psi(x,\epsilon) \to \infty$ as $x \to 0$. Hence $\langle \tan^{-1} nx \rangle$ is Not u.c on $[0,1]$

Now if we take $[a,1]$ & $a>0$ then

$$\sup_{x \in [a,1]} \psi(x,\epsilon) = \frac{\cot \epsilon}{a}$$

then $n > \left[\frac{\cot \epsilon}{a} \right] = m$ [depend only on ϵ]

So $\langle \tan^{-1} nx \rangle$ is u.c on $[a,1]$ $a>0$ and u.c on every that interval which doesn't contain zero.

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Que $\langle f_n(x) \rangle = \left\langle \frac{x}{n+x} \right\rangle$ is u.c on $[0, b]$. $b<\infty$.

But only Pointwise cgt when the interval extend to ∞.

Soln:- For limit function, Let $x \in [0,b]$, ~~$b<\infty$~~

point seqn $\langle f_n(x) \rangle = \left\langle \frac{x}{n+x} \right\rangle \to 0$ as $n \to \infty$

Hence $f(x) = 0 \ \forall \ x \in \cancel{\mathbb{R}} [0,b]$.

Now $|f_n(x) - f(x)| = \left| \frac{x}{n+x} - 0 \right| < \epsilon$

$n > x\left(\frac{1}{\epsilon} - 1\right)$

$\sup_{x\in[0,b]} \psi(x,\epsilon) = b\left(1/\epsilon - 1\right)$, but $b<\infty$.

$n > \left[b\left(1/\epsilon - 1\right)\right] = m$ [depends only ϵ, Not on x]

Hence, $\left\langle \frac{x}{n+x} \right\rangle$ is U.C on $[0,b]$, $b<\infty$.

And Not U.C on $[0,\infty)$ or $[0,\infty]$

Mn - Test :- Let $\langle f_n(x)\rangle$ be a sequence of functions on I and $f(x)$ be its limit function i.e.

$\lim_{n\to\infty} f_n(x) = f(x)$

Now define $M_n = \sup_{x\in I} \{|f_n(x) - f(x)|\}$.

M_n = (function of n only).

then $\langle f_n(x)\rangle$ is uniformly cgt iff $\lim_{n\to\infty} M_n = 0$

Note :- This test is very strong because it is Necessary and sufficient if $\lim_{n\to\infty} M_n \neq 0$ then $\langle f_n(x)\rangle$ is Not U.C.

Qu $\langle x^n\rangle$, $x\in[0,1]$.

$$M_n = \sup_{x\in[0,1]} \{|x^n - 0|\}$$
$$= \sup_{x\in[0,1]} |x^n| = 1$$

(∵ x^n is strictly increasing so $\sup x^n$ will give at $x \to 1$)

Hence , $M_n = 1$ so $\lim_{n\to\infty} M_n \neq 0$

Hence $\langle x^n \rangle$ is Not u.c on $[0,1]$.

if interval $[0,a]$, $a<1$, then $M_n = a^n$
then $\lim_{n\to\infty} a^n = 0$

Hence $\langle x^n \rangle$ is u.c on $[0,a]$, $a<1$.

Que $f_n(x) = \begin{cases} -n^2x & ; \quad 0 \le x < 1/n \\ -n^2x + 2n & ; \quad 1/n \le x < 2/n \\ 0 & ; \quad 2/n \le x \le 1 \end{cases}$

Limit function $f(x) = 0 \quad \forall \ x \in [0,1]$

Now $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$

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$M_n = \left\{ \sup_{x\in[0,1/n)} |f_n(x) - f(x)| , \sup_{x\in[1/n,2/n)} |f_n(x) - f(x)| , \sup_{x\in[2/n,1]} |f_n(x) - f(x)| \right.$

$= \left\{ \sup_{x\in[0,1/n)} |n^2x| , \sup_{x\in[1/n,2/n)} |-n^2x+2n| , \sup_{x\in[2/n,1]} |0-0| \right.$

$= \left\{ |n^2 \times \frac{1}{n}| ; |-n^2 \times \frac{1}{n} + 2n| , 0 \right\}$

[∵ n^2x is ↑ so Max at $x = 1/n$
$-n^2x + 2n$ is ↓ so Max at $1/n$].

$M_n = \{n, n, 0\}$

$M_n = n$

Hence $\lim\limits_{n\to\infty} M_n = \infty \neq 0$ so

$\langle f_n(x)\rangle$ Not u.c on $[0,1]$.

Q4 $\langle f_n(x)\rangle = \langle \tan^{-1} nx\rangle$ $x \in [0,1]$.

$$f(x) = \begin{cases} 0 & , x = 0 \\ \pi/2 & , \text{else} \end{cases}$$

$$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\tan^{-1}nx - \frac{\pi}{2}\right|$$

$$= \sup_{x\in[0,1]} |\cot^{-1}nx| \quad [\because \cot^{-1}nx \text{ is } \downarrow \text{ so Max at } x \to 0$$

$M_n = \frac{\pi}{2}$ So $\lim\limits_{n\to\infty} M_n \neq 0$

$\Rightarrow \langle f_n(x)\rangle$ is Not u.c on $[0,1]$.

if interval $[0,1]$; $a > 0$

then $M_n = \cot^{-1} na$ So $\lim\limits_{n\to\infty} M_n = 0$

$\Rightarrow \langle \tan^{-1} nx\rangle$ is u.c on $[a,1]$, $a > 0$

Q4 $\langle f_n(x)\rangle = \left\langle \dfrac{nx}{1+n^2x^2}\right\rangle$; $x \in \mathbb{R}$.

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$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall\ x \in \mathbb{R}.$$

Now $M_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$

$$= \sup_{x\in\mathbb{R}} \left| \frac{nx}{1+n^2x^2} - 0 \right|$$

$$= \sup_{x\in\mathbb{R}} \left| \frac{nx}{1+n^2x^2} \right| \quad \text{--- (1)}$$

let $y = \dfrac{nx}{1+n^2x^2}$

$y' = \dfrac{n - n^3x^2}{(1+n^2x^2)^2}$, Put $y' = 0$

$\Rightarrow x = \pm \dfrac{1}{n}$

then y has max at $1/n$

So $\left| \dfrac{nx}{1+n^2x^2} \right|$ has max at $\pm \dfrac{1}{n}$

but $x = \pm \dfrac{1}{n}$ in (1) then

$M_n = \dfrac{1}{2}$, Hence $\lim_{n\to\infty} M_n \neq 0$, so

$\left\langle \dfrac{nx}{1+n^2x^2} \right\rangle$ is Not u.c on $\mathbb{R}$ or on $[0, \infty)$.

Que $\langle f_n(x) \rangle = \left\langle \dfrac{n^2x}{1+n^2x^2} \right\rangle$ on $[0,1]$

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1/x, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

$|f_n(x) - f(x)| = \left| \frac{n^2 x}{1+n^2x^2} - \frac{1}{x} \right| = \frac{1}{x(1+n^2x^2)} < \epsilon$

$\Rightarrow 1 < \epsilon \cdot x(1+n^2x^2)$

$\Rightarrow x \cdot (1+n^2x^2) > \frac{1}{\epsilon}$

$n > \frac{1}{x}\sqrt{\frac{1}{\epsilon x} - 1}$ $\left(\because 0 < x \leq 1 \Rightarrow 0 < \frac{1}{x} \leq 1 \right)$

Here as $x \to 0$, $\frac{1}{x}\sqrt{\frac{1}{\epsilon x} - 1} \to \infty$, so it is

Not possible to choose a +ve integer m s.t.

$|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq m, \ \forall\ x \in [0,1]$

Hence $\langle f_n(x) \rangle$ is Not U.C on $[0,1]$.

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Q4 $\langle f_n(x) \rangle = \left\langle \frac{nx}{1+n^3x^2} \right\rangle$, $x \in \mathbb{R}$,

$f(x) = 0 \quad \forall\ x \in \mathbb{R}$

Now $M_n = \sup_{x \in \mathbb{R}} \left| \frac{nx}{1+n^3x^2} \right|$

Let $y = \frac{nx}{1+n^3x^2}$

Put $y' = 0$

$\Rightarrow n - n^4x^2 = 0$

$x = \pm 1/n^{3/2}$

y will give Max at $x = \frac{1}{n^{3/2}}$. Hence Put $x = \frac{1}{n^{3/2}}$

then $M_n = \frac{1}{2\sqrt{n}}$, $\lim_{n\to\infty} M_n = 0$

⇒ $\langle f_n(x) \rangle$ is U.C ~~is~~ on IR

Qu $\langle f_n(x) \rangle = \left\langle \frac{x}{1+nx} \right\rangle$, $f(x) \equiv 0 \quad \forall \; x \in [0,\infty)$

$$M_n = \sup_{x\in[0,\infty)} \left| \frac{x}{1+nx} - 0 \right| = \sup_{x\in[0,\infty)} \left| \frac{x}{1+nx} \right|$$

let $y = \frac{x}{1+nx}$; $y' = \frac{1}{(1+nx)^2} > 0 \;\; \forall \; x \; [0,\infty)$.

Here $y' > 0$ so y is strictly increasing, Hence y will have max. at $x \to \infty$. so

$$M_n = \frac{1}{n} \left[\lim_{x\to\infty} y = \frac{1}{n} = \sup \frac{x}{1+nx} \right.$$

Now $\lim_{n\to\infty} M_n = 0$

⇒ $\langle f_n(x) \rangle$ is U.C on $[0,\infty)$.

Ex:- $\langle f_n(x) \rangle = e^{-nx}$ on $[0,\infty)$.

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x > 0 \end{cases}$$

$$M_n = \sup_{x\in(0,\infty)} |e^{-nx}-0| = \sup_{x\in(0,\infty)} |e^{-nx}|$$

let $y = e^{-nx}$

$$y' = \frac{-n}{e^{nx}} < 0 \;\forall\; x \in (0,\infty).$$

y is s.↓. Hence y will give Max at $x \to 0$

Hence, $M_n = 1$ so $\lim\limits_{n\to\infty} M_n \neq 0$

$\langle f_n(x)\rangle = \langle e^{-nx}\rangle$ is Not U.C on $[0,\infty)$

Now, if interval $[a,\infty)$ $a>0$ then y will give Max at $x=a$, Now

$$M_n = e^{-na}, \quad \lim_{n\to\infty} M_n = 0$$

$\Rightarrow \langle e^{-nx}\rangle$ is U.C on $[a,\infty)$, $a>0$

Qu $f_n(x) = \begin{cases} |nx-1|, & 0 \le x < 2/n \\ 1, & 2/n \le x \le 1 \end{cases}$

$\langle f_n(x)\rangle$ can be defined as –

$$f_n(x) = \begin{cases} 1-nx, & 0 \le x < 1/n \\ nx-1, & 1/n \le x < 2/n \\ 1, & 2/n \le x \le 1 \end{cases}$$

$\lim\limits_{n\to\infty} f_n(1) = 1 \;\; \forall \;\; x \in [0,1]$

$$M_n = \sup \begin{cases} \sup |f_n(x)-f(x)| ; & x \in [0, 1/n) \\ \sup |f_n(x)-f(x)| ; & x \in [1/n, 2/n) \\ \sup |f_n(x)-f(x)| ; & x \in [2/n, 1] \end{cases}$$

$$= \sup \begin{cases} \sup |1-nx-1| , & x \in [0, 1/n) \\ \sup |nx-1-1| , & x \in [1/n, 2/n) \\ \sup |1-1| ; & x \in [2/n, 1] \end{cases}$$

$$= \sup \begin{cases} 1 \\ 0 \\ 0 \end{cases}$$

$M_n = 1$

$\lim_{n\to\infty} M_n \neq 0$

$\Rightarrow$ $\langle f_n(x) \rangle$ is Not U.C on $[0,1]$.

Q-10 $\langle f_n(x) \rangle = \langle n x e^{-nx^2} \rangle$ $x \in (0,\infty)$

$\lim_{n\to\infty} f_n(x) = f(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{x}{+x^2 \cdot e^{+nx^2}} = 0$

Hence $\boxed{f(x) = 0}$ $\forall$ $x \in (0,\infty)$

$M_n = \sup_{x\in(0,\infty)} |nxe^{-nx^2} - 0| = \sup_{x\in(0,\infty)} |nxe^{-nx^2}|$

let $y = n \cdot xe^{-nx^2}$

$y' = n \cdot e^{-nx^2} [1 - 2nx]$

$y' = 0$

$x = \pm \frac{1}{}$

Y will give Max at $x = \frac{1}{\sqrt{2n}}$, Hence

$$M_n = n \times \frac{1}{\sqrt{2n}} \cdot e^{-n \cdot \frac{1}{2n}} = \frac{\sqrt{n}}{2} e^{-1/2}$$

$$M_n = \frac{\sqrt{n}}{2\sqrt{e}} \; ; \; \lim_{n\to\infty} M_n = \infty \neq 0$$

Hence $\langle nxe^{-nx^2} \rangle$ [illegible] $(0,\infty)$

Q+11 $\langle f_n(x) = \frac{x^n}{n} \rangle$, $x \in (-\infty, \infty)$

$$\lim_{n\to\infty} f_n(x) = \; = f(x) = \begin{cases} \infty & |x| > 1 \\ 0 & |x| \leq 1 \end{cases}$$

Here P.W.L does Not exist $\forall$ $x \in (-\infty, \infty)$ so $\langle f_n(x) \rangle$ is Not u.c on $(-\infty, \infty)$ but if we take interval $[-1,1]$ then by M_n - test.

$$M_n = \sup_{x\in[-1,1]} \left| \frac{x^n}{n} - 0 \right| = \sup_{x \in [-1,1]} [illegible]$$

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Let $y = \frac{x^n}{n}$ [illegible] $y' = 0 \Rightarrow x = 0$

Here $y'' = 0$ at $x = 0$, So [illegible] can't discuss Max Y by this Method.

Now, as we know $\left| \frac{x^n}{n} \right|$ is sequence of

$\langle x, \frac{x^2}{2}, \frac{x^3}{3}, \ldots \rangle$

So all this function will give it's Max at $x = 1$

So y has Max value at $x=1$.

So $M_n = \frac{(1)^n}{n} = \frac{1}{n}$

Hence $\lim_{n\to\infty} M_n = 0 \Rightarrow \langle \frac{x^n}{n} \rangle$ is u.c on $|x| \le 1$

Th Properties of Uniform convergence :-

If sequence of functions $\langle f_n(x) \rangle$ converges to $f(x)$ uniformly on $[a,b]$ then.

(i) $\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x) = \lim_{x\to a} f(x)$ provided $\lim_{x\to a} f(x)$ exist.

(ii) if each $f_n(x)$ is cts then $f(x)$ is cts. i.e $f_n(x)$ is cts $\forall\, n \in \mathbb{N} \Rightarrow f(x)$ is cts.

Note :- if each $f_n(x)$ is cts but $f(x)$ is Not cts then $\langle f_n(x) \rangle$ can't uniformely cgt.

Eg: $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$; $x \in \mathbb{R}$

$$f(x) = \begin{cases} 1 & |x| > 1 \\ 1/2 & |x| = 1 \\ 0 & |x| < 1 \end{cases}$$

each $f_n(x)$ is cts but limit function $f(x)$ is discts on $\mathbb{R}$. Hence $\langle f_n(x) \rangle$ is Not u.c on $\mathbb{R}$.

3) If each $f_n(x)$ is differentiable then $f(x)$ may Not be differentiable.

4) If each $f_n(x)$ is differentiable and seqn $\langle f_n'(x)\rangle$ converges uniformely to $g(x)$ then [illegible] is diff and $f'(x)=g(x)$

Note Each f_n [illegible] differentiable but it is quite posible that $\langle f_n'(x)\rangle$ may Not Ato Converge even pointwise through $\langle f_n(x)\rangle$ converges un uniformly to f.

⑤ If each $f_n(x)$ is R- integrable i.e $f_n(x) \in R[a,b]$ $\forall$ $n \in \mathbb{N}$ then f is R- integrable. and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x) = \int_a^b f(x)\,dx.$$

Note If each $f_n \in R[a,b]$ and [illegible]

$\lim_{n\to\infty} \int_a^b$ [illegible] $\lim_{n\to\infty} f_n(x)$ [illegible] then $\langle f_n(x)\rangle$

Can't be u.c.

Eg:- $f_n(x) = \begin{cases} n^2x & ; \ 0 \le x < 1/n \\ -n^2x+2n & ; \ 1/n \le x < 2/n \\ 0 & ; \ 2/n \le x \le 1 \end{cases}$

Now $\int_0^1 f_n(x)\,dx = \int_0^{1/n} n^2x\,dx + \int_{1/n}^{2/n} (-n^2x+2n)\,dx + \int_{2/n}^{1} 0\,dx$

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$\Rightarrow \int_0^1 fn(x) dx = 1$

So $\lim_{n\to\infty} \int_0^1 fn(x) dx = 1$

$\because$ limit function $f(x) = 0$ so.

$\int_0^1 fn(x) dx = \int_0^1 f(x) dx = \int_0^1 0 \, dx = 0$

Hence each fn(x) is R- integrable on [0,1]

but $\lim_{n\to\infty} \int_0^1 fn(x) dx \neq \int_0^1 \lim_{n\to\infty} fn(x) dx.$

$\Rightarrow$ < fn(x) > is Not U.C on [0,1].

Note :- Each Property from 1 to 5 only Sufficient condition. So use all these carefully.

Some other Example :-

A) A seq^n of cts functions with discts limit function.

E.g.:- $fn(x) = x^{1/n}$ $x \in [0,1]$, each fn(x) is cts.

but $f(x) = \begin{cases} 0 & ; x = 0 \\ 1 & ; \text{else} \end{cases}$ is discts.

B A sequence for which $\lim_{n\to\infty} \int_0^1 fn(x) dx \neq \int_0^1 \lim_{n\to\infty} fn(x)$.

E.g:- $\langle fn(x) \rangle = n^2 x (1-x)^n$; $x \in [0,1]$

$f(x) = 0 \quad \forall \quad x \in [0,1]$

So $\int_0^1 f(x)\,dx = 0.$

And. $\int_0^1 f_n(x)\,dx = n^2 \int_0^1 x(1-x)^n\,dx$

$= n^2 \left[\frac{-x\cdot(1-x)^{n+1}}{n+1} - \frac{(1-x)^{n+2}}{(n+1)(n+2)} \right]_0^1$

$n^2\left[\frac{1}{(n+1)(n+2)}\right]$

$\int_0^1 f_n(x)\,dx = \frac{n^2}{(n+1)(n+2)}$

Now, $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$

Here $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)$

i'e $\lim_{n\to\infty} \int_0^1 f_n(x) \neq \int_0^1 f(x)\,d$

c) The limit of differentiation is Not equal to differentiation of limit

Eg: $\langle f_n(x)\rangle = \left\langle \frac{\sin nx}{\sqrt{n}} \right\rangle \quad x \in [0,\pi]$.

$\lim_{n\to\infty} \frac{1}{(n)^{1/2}}\cdot \sin nx = 0 = f(x) \;\forall\; x \in [0,\pi]$

Now $f_n'(x) = \sqrt{n}\cdot\cos nx$ & Here

$\lim_{n\to\infty} f_n'(x) = \sqrt{n}\cdot\cos nx$, Here

$\lim_{n\to\infty} fn'(x)$ does not exist for any $x \in [0,\pi]$.

Let $x=0$ then $fn'(0) = \sqrt{n} \to \infty$ as $n \to \infty$.

Hence $f'(0) = 0$ and $fn'(0) \to \infty$ as $n \to \infty$

Hence, $\boxed{\lim_{n\to\infty} fn'(0) \neq f'(0)}$

NET
Q4 $fn(x) = \begin{cases} 1-nx ; & x \in [0, 1/n) \\ 0 ; & x \in [1/n, 1] \end{cases}$ then

a) $\lim_{n\to\infty} f_n(x)$ defines a cts function.

b) $\langle f_n(x) \rangle$ U.C on $[0,1]$.

c) ✓ $\lim_{n\to\infty} fn(x)$ exists for all $x \in [0,1]$

d) $\lim_{n\to\infty} fn(x) = 0 \ \forall \ x \in [0,1]$.

Here $f(x) = \begin{cases} 1, & x=0 \\ 0, & \text{else} \end{cases}$ is Not cts.

∴ each $fn(x)$ is cts but $f(x)$ is discts so $fn(x)$ is Not U.C on $[0,1]$.

$\lim_{n\to\infty} f_n(x)$ exists $\forall \ x \in [0,1]$ but Not identically

zero.

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NET Qu

$f_n(x) = x^{1/n}$ $x \in [0,1]$ then

(a) $\lim_{n\to\infty} f_n(x)$ defines a cts function

(b) $\lim_{n\to\infty} f_n(x)$ exists $\forall$ $x \in [0,1]$.

(c) $\langle f_n(x)\rangle$ is [illegible]

(d) $\lim_{n\to\infty} f_n(x) =$ [illegible] $[0,1]$ [same as Previous]

$$f(x) = \begin{cases} 0 & ; \quad x=0 \\ 1 & ; \quad \text{else} \end{cases}$$

$f(x)$ is Not cts.

$\Rightarrow$ $f_n(x)$ is Not u.c.

and $\lim_{n\to\infty} f_n(x) = f(x)$ exist $\forall$ [illegible] But

Not $\lim_{n\to\infty} f_n(x) = 0$ [illegible] $x \in [0,1]$

# Let $D \subseteq \mathbb{R}$, $f_n : D \longrightarrow \mathbb{R}$ & $\langle f_n\rangle$ seq^n of function. A seq^n of function is uniformly bdd. and seq^n $\langle f_n\rangle$ is uniformly cgt to f then f is bdd.

$$\begin{tikzcd} \langle f_n\rangle \arrow[r, "U.C"] \arrow[d] & f \arrow[d] \\ \text{uniformly bdd} & \text{bdd.} \end{tikzcd}$$

if fn is pointwise cgt then we cannot say f is bdd.

② Let $D \subseteq \mathbb{R}$, $f_n : D \longrightarrow \mathbb{R}$, $\langle f_n(x) \rangle$ be the seqn of cts function and fn is uoc to limit function f then f is cts.

if fn is pointwise convergent then we cannot say f is cts.

NET
# Let D is compact set of $\mathbb{R}$. $f_n : D \longrightarrow \mathbb{R}$ fn is seqn of integrable fuction and fn is uoc to f (limit function). then f is integrable..

if fn is pointwise cgt then we cannot say f is integrable.

Dini theorem :- Let D is compact subset of $\mathbb{R}$. $f_n : D \longrightarrow \mathbb{R}$ be seqn of which is pointwise convergent to continuous function f. and $f_n(x)$ is Monotone function for each x then seqn fn is u.c.

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SERIES OF FUNCTIONS :-

Let $\langle f_n(x)\rangle$ be a sequence of fuctions on I, then $\sum f_n(x)$ is called series of Real valued Functions.

i.e $\sum f_n(x) = f_1(x) + f_2(x) + --- + f_n(x) + ---$

Now, $S_1(x) = f_1(x)$

$S_2(x) = f_1(x) + f_2(x)$

$S_3(x) = f_1(x) + f_2(x) + f_3(x)$

$\vdots$

$S_n(x) = f_1(x) + f_2(x) + f_3(x) + --- + f_n(x)$

$\vdots$

then sequence $\langle S_n(x)\rangle$ is called sequence of partial sums (SOPS) of $\sum f_n(x)$. $S_1(x), S_2(x), ---, S_n(x), ---$ are called the first, second, ---, nth term of $\langle S_n(x)\rangle$

Behaviour of Series :-

A series of functions $\sum f_n(x)$ converges pointwise or uniformely according as its SOPS $\langle S_n(x)\rangle$ converges pointwise or uniformely.

Eg:- (i) $\sum f_n(x) = \sum\left(\frac{nx}{1+n^2x^2} - \frac{(n-1)\cdot x}{1+(n-1)^2x^2}\right)$

SOPS $\langle S_n(x)\rangle = f_1(x) + f_2(x) + --- + f_n(x)$

$= \left(\frac{1}{1+x^2} - 0\right) + \left(\frac{2x}{1+4x^2} - \frac{x}{1+x^2}\right) + --- \left(\frac{nx}{1+n^2x^2} -\right.$

$$- \frac{(n-1)\cdot x}{1+(n-1)\cdot x^2} \Big)$$

$$\Rightarrow \langle S_n(x) \rangle = \left\langle \frac{nx}{1+n^2x^2} \right\rangle$$

Seqn $\langle S_n(x) \rangle = \frac{nx}{1+[illegible]}$ $\forall\, x \in \mathbb{R}$

Limit function $f(x) = [illegible]$ $\forall\, x \in \mathbb{R}$.

By M_n – Test

$$M_n = \frac{1}{2}$$

$$\lim_{n\to\infty} M_n \neq 0$$

$\Rightarrow$ $\langle S_n(x) \rangle$ is only pointwise convergent Not U.C

$\Rightarrow$ $\sum f_n(x)$ is only pointwise convergent Not uniformely.

Q) $$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} x \cdot e^{-nx}, \quad x \in [0,1]$$

$$S_n(x) = f_1(x) + f_2(x) + \dots + f_n(x)$$

$$= x\, e^{0} + x e^{-1\cdot x} + \dots + x e^{-n x}$$

$$= x \Big[1 + [illegible] + \dots + \frac{1}{e^{nx}} \Big]$$

G.P.

$$= x \left[\frac{1 \cdot (1 - 1/e^{nx})}{1 - 1/e^{x}} \right]$$

$$= \frac{x \cdot e^{x}}{e^{x}-1} \left[1 - \frac{1}{e^{nx}} \right]$$

Now limit function $f(x) = \begin{cases} 0 & ; x = 0 \\ \frac{x \cdot e^{x}}{[illegible]} & ; x \in (0,1] \end{cases}$

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consider $x \in [0,1]$, Apply M_n - test.

$$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| , \quad x \in (0,1] .$$

$$= \sup_{x\in[0,1]} \left| \frac{x\cdot e^x}{e^x-1}\left(1-\frac{1}{e^{nx}}\right) - \frac{x\cdot e^x}{e^x-1} \right|$$

$$= \sup_{x\in[0,1]} \left| \frac{x\cdot e^x}{(e^x-1)\, e^{nx}} \right|$$

let $y = \frac{x\cdot e^x}{(e^x-1)e^{nx}}$, $y' = 0 \Rightarrow x = \frac{1}{n} \in (0,1]$.

y will take its Max value at $x = 1/n$.

Hence

$$M_n = \frac{1/n \cdot e^{1/n}}{(e^{1/n}-1)\cdot e} \; ; \quad \lim_{n\to\infty} M_n = \frac{1}{e} \neq 0$$

$\Rightarrow \langle S_n(x) \rangle$ is Not U.C on $[0,1]$

$\Rightarrow \sum_{n=0}^{\infty} x\cdot e^{-nx}$ is Not U.C on $[0,1]$

Some another tests to show $\sum f_n(x)$ is uniformely convergent

7) Weierstrass's M - test :- Let $\sum f_n(x)$ be a given series, then $\sum f_n(x)$ will converges uniformely on I if there exists a convergent series $\sum_1^{\infty} M_n$ s.t

$$|f_n(x)| \le M_n \;\; \forall\, n \in \mathbb{N} \;\; \& \;\; \forall\, x \in I$$

Note :- If you are Not able to find such convergent

Series $\sum M_n$ or you find $\sum M_n$ which is not convergent and satisfied $|f_n(x)| \le M_n$ then it doesn't implies ~~[illegible]~~ that $\sum f_n(x)$ is not U.C. i.e Test is applicable only if $\sum M_n$ is conv. otherwise test fail.

Eg (i) $\sum f_n(x) = \sum \frac{\sin nx}{n^2}$ [illegible]

Here $\left|\frac{\sin nx}{n^2}\right| \le \left|\frac{1}{n^2}\right|$

& $\sum \frac{1}{n^2}$ is conv $\Rightarrow \sum \frac{\sin nx}{n^2}$ is U.C on $\mathbb{R}$,

(ii) $\sum \frac{\sin nx}{n^\alpha}$, $\alpha > 0$, $\alpha \in \mathbb{R}$.

$$\left|\frac{\sin nx}{n^\alpha}\right| \le \left|\frac{1}{n^\alpha}\right|$$

Here $\sum M_n = \sum \frac{1}{n^\alpha}$ conv if $\alpha > 1$
div if $\alpha \le 1$

$\Rightarrow \sum \frac{\sin nx}{n^\alpha}$ U.C on $\mathbb{R}$ if $\alpha > 1$

and have no Result through M-test because for $\alpha \le 1$ $\sum M_n$ is div.

(iii) $\sum_{1}^{\infty} f_n(x) = \sum_{1}^{\infty} \frac{1}{1+n^2 x}$ in $[1, \infty)$

$$|f_n(x)| = \left|\frac{1}{1+n^2 x}\right| \le \left|\frac{1}{n^2}\right|$$

and $\sum \frac{1}{n^2}$ is cgt $\Rightarrow \sum_{1}^{\infty} \frac{1}{1+n^2 x}$ is u.c in $[1, \infty)$

④ $\sum f_n(x) = \sum_{1}^{\infty} \frac{x}{(n+x^2)^2}$, $x \in \mathbb{R}$

$f_n(x) = \frac{x}{(n+x^2)^2}$

Now $f_n'(x) = 0 \Rightarrow (n+x^2)^2 - 4x^2(n+x^2) = 0$

$\Rightarrow 3x^4 + 2nx^2 - n^2 = 0$

$\Rightarrow x^2 = \frac{n}{3}$

$\Rightarrow x = \sqrt{\frac{n}{3}}$

and $f_n''(x) < 0$ at $x = \sqrt{\frac{n}{3}}$

Hence $f_n(x)$ at $x = \sqrt{\frac{n}{3}} = \frac{\sqrt{n/3}}{(n+n/3)^2} = \frac{3\sqrt{3}}{16\, n^{3/2}} = M_n$

$\sum M_n$ is cgs $\Rightarrow \sum \frac{x}{(n+x^2)^2}$ is u.c.

(5) $\sum f_n(x) = \sum \frac{x}{n(1+nx^2)}$, $x \in \mathbb{R}$ (I.A.S)

is u.c. [Do Yourself]

Hint :- $\left\{ f_n'(x) = 0 \Rightarrow x = 1/\sqrt{n}\right.$, Put $x = \frac{1}{\sqrt{n}}$ For Max. value of $f_n(x)$, then $\left. M_n = \frac{1}{2 \cdot n^{3/2}} \right\}$

Defn:- Uniformely Bounded :- Sequence of functions $\langle f_n(x) \rangle$ is said to be uniformely Bounded if $\exists$ M s.t.

$$|f_n(x)| \leq M \quad \forall\ x \in I \text{ and } \forall\ n \in \mathbb{N}.$$

Note:- Seqn $\langle f_n(x) \rangle$ is uniformely Bounded if set of Bounds (lower and upper) of the member of family is bounded.

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② **Dirichlet's Test:-** (More stranger than weierstrass)

if $\sum f_n(x)$ can be written as Product of two functions $u_n(x)$ and $v_n(x)$ i.e $\sum f_n(x) = \sum u_n(x) \cdot v_n(x)$ s.t.

i) $v_n(x)$ is Monotonically decreasing and converges to zero uniformely.

ii) The SOPS of $\sum u_n(x)$ is uniformely Bounded, then $\sum u_n(x) \cdot v_n(x)$ is uniformly convergent.

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Eg:- $\sum f_n(x) = \sum \frac{(-1)^{n-1}}{n+x^2}$, $x \in \mathbb{R}$

Let $u_n(x) = (-1)^{n-1}$, $v_n(x) = \frac{1}{n+x^2}$

Here $v_n(x)$ is $\downarrow$ and $\lim_{n\to\infty} v_n(x) = 0 \ \forall\ x$.

and SOPS of $\sum u_n(x) = 1 + (-1) + 1 + (-1) + \cdots + (-1)^{n-1}$

$$= \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$$

i.e $\sum_{r=1}^{n} u_r(x)$ = SOPS of $\sum u_n(x)$ is bdd by 0 & 1.

Hence both conditions hold, then.

$$\sum u_n(x) \cdot v_n(x) = \sum \frac{(-1)^n}{n+x^2} \text{ is U.C on } \mathbb{R}.$$

② $\sum f_n(x) = \sum \frac{x^n}{n+1}$, $x \in [-a,a]$, $a<1$

Let $u_n(x) = x^n$, $v_n(x) = \frac{1}{n+1}$

Here $v_n(x)$ is ↓ & $v_n(x) \to 0$ as $n \to \infty$

and SOPS of $\sum u_n(x)$

$$= |x|+|x|^2+|x|^3+ --- +|x|^n$$
$$\le a+a^2+a^3+ --- +a^n \quad (G.P)$$
$$= \frac{a(1-a^n)}{1-a}$$
$$< \frac{a}{1-a}$$

→ SOPS of $\sum u_n(x)$ is uniformly bdd.

⇒ $\sum f_n(x) = \sum \frac{x^n}{n+1}$ is U.C on $[-a,a]$, $a<1$.

③ $\sum f_n(x) = \sum (-1)^{n-1} x^n$, $x \in [0,k]$, $k<1$

Here, $v_n(x) = x^n$ ↓ and $\lim_{n\to\infty} x^n = 0$ as $x \in [0,k]$

and S.O.P.S of

$$\sum (-1)^n = \begin{cases} 0 & \text{when } n \text{ is even} \\ 1 & \text{when } n \text{ is odd} \end{cases}$$

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$\Rightarrow$ $\sum (-1)^n . x^n$ is u.c on $[0, k]$, $k<1$.

④ $\sum f_n(x) = \sum \frac{\sin nx}{n^\alpha}$, $\alpha > 0$; $0 < a \le x < b < \frac{\pi}{2}$

Let $\langle v_n(x) \rangle = \langle \frac{1}{n^\alpha} \rangle$ is $\downarrow$ and $\to 0$ as $n \to \infty$

$\langle$ S.O.P.S $\rangle$ $S_n(x) = \sum \sin nx$

$= \sin x + \sin 2x + \cdots + \sin nx$

$$= \left| \frac{\sin\frac{(n+1)x}{2} \cdot \sin\frac{nx}{2}}{\sin\frac{x}{2}} \right| \le \left| \frac{1}{\sin\frac{x}{2}} \right|$$

$\because$ $\sin\frac{x}{2}$ is $\uparrow$ is $(0, \pi/2)$ & $\frac{1}{\sin x/2}$ $\downarrow$ in $(0, \pi/2)$

then $|S_n(x)| \le \left| \frac{1}{\sin x/2} \right|$

$\Rightarrow$ $\langle S_n(x) \rangle$ is uniformly bdd

Hence $\sum \frac{\sin nx}{n^\alpha}$ is u.c $\forall$ $\alpha > 0$ & $\forall$ $x \in (a, b]$

$a > 0$ & $b < \pi/2$.

⑤ $\sum f_n(x) = \sum (-1)^n . \frac{1}{n}$ (Do yourself)

⑥ $\sum f_n(x) = \sum \frac{\sin n}{\sqrt{n}}$ (Do yourself)

~~⑦ $\sum f_n(x) = \sum (-1)^n \cos\frac{1}{n}$ (")~~

Note:- Leibnitz test for alternating series of Real Numbers is Particular Case of Dirichlet's test.

Ques:- If $\sum a_n$ is Absolutely convergent of Real Numbers then show that $\sum \frac{x^{2n} a_n}{1+x^{2n}}$ is uniformly cgt.

Soln. $\because \left|\frac{x^{2n}}{1+x^{2n}}\right| \le 1$

then $\left|a_n \cdot \frac{x^{2n}}{1+x^{2n}}\right| \le |a_n|$

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$\because$ $\sum a_n$ is Abosolutely cgt., then

$\sum \frac{x^{2n}}{1+x^{2n}} \cdot a_n$ is Abosolutely cgt, Hence U.C.

③ **Abel's Test :-** The Series $\sum u_n(x) \cdot v_n(x)$ will be uniformely convergent if

I) $\sum u_n(x)$ is uniformely convergent in $[a,b]$

II) Seqn $\langle v_n(x) \rangle$ is Mononic decreasing and $\langle v_n(x) \rangle$ is Uniformely bdd in $[a,b]$ then

$\sum f_n(x) = \sum u_n(x) \cdot v_n(x)$ is uniformely cgt.

Eg:- $\sum f_n(x) = \sum \frac{(-1)^{n-1}}{n} x^n$, $x \in [0,1]$

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Take $v_n(x) = x^n$ & $u_n(x) = \frac{(-1)^{n-1}}{n}$.

Here $\langle v_n(x) \rangle$ is uniformly bdd & Monotonic decreasing.

and $\sum u_n(x) = \sum \frac{(-1)^n}{n}$ is cgt [by Leibnitz test].

Hence $\sum$ [illegible] is U.C on [0,1]

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② If $\sum a_n$ is convergent then what about

$\sum \frac{a_n}{n^x}$; $x \in [0,1]$.

Let $\sum f_n(x) = \sum \frac{a_n}{n^x}$

Take $\sum u_n(x) = \sum a_n$ is cgt &

$\langle v_n(x) \rangle = \left\langle \frac{1}{n^x} \right\rangle$ decreasing and uniformly bdd on [0,1] i.e $|v_n(x)| \le \left|\frac{1}{n^x}\right| \le 1$

Hence, by Abel's test $\sum \frac{a_n}{n^x}$ is U convergent.

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③ $\sum (-1)^n \cdot \frac{x}{1+nx}$, $x \in [0,1]$. (Do yourself).

Properties :- If $\sum f_n(x)$ is uniformly cgt and converges to $f(x)$ [$f(x)$ is sum of functions], then

1) If each term is cts function then sum of functions f(x) is also cts.

2) $\lim_{x\to a} \sum f_n(x) = \sum \lim_{n\to\infty} f_n(x) = \lim_{x\to a} f(x)$.

Eg:- Prove that $\lim_{x\to 1}\left[\sum_{1}^{\infty} \frac{nx^2}{n^3+x^3}\right] = \sum_{1}^{\infty} \frac{n}{n^3+1}$

∵ $\sum \frac{nx^2}{n^3+x^2}$ is u.c on $[0, k]$ $k>0$ by Abel's Test. Hence $\sum$ and limit can be interchanged. then.

$$\lim_{x\to 1}\left[\sum_{1}^{\infty} \frac{nx^2}{n^3+x3}\right] = \sum_{1}^{\infty}\left[\lim_{x\to 1} \frac{nx^2}{n^3+x^3}\right] = \sum_{1}^{\infty} \frac{n}{n^3+1}$$

3) if each term is integrable then sum of function f(x) is integrable and

$$\int \sum f_n(x) = \sum \int f_n(x) = \int f(x)dx$$

i.e we can interchange the sign of integration and summation.

① $\sum f_n(x)$ is converges uniformly to f(x) and $f_n(x)$ is differentiable for each $n \in \mathbb{N}$, and

$\sum f_n'(x)$ converges uniformly to $g(x)$ then $f'(x) = g(x)$

Note:- If $\sum f_n(x)$ is converges uniformly to $f(x)$ and each $f_n(x)$ is differentiable then $f(x)$ Need Not ~~to~~ be diff. (It may be diff or may Not)

Some other Questions :-

① $\sum r^n \cos n\theta$, $\theta \in \mathbb{R}$, $0 < r < 1$

Apply Weierstrass M-test

$|r^n \cos n\theta| \leq |r^n|$ $\forall$ $n \in \mathbb{N}$ Here $M_n = r^n$.

Now $\sum r^n$ is cgs [common Ration $r < 1$] $|r|$

let $\langle S_n \rangle$ be its SOPS $[\sum r^n]$.

$$S_n = r + r^2 + \ldots + r^n$$

$$= r\left[\frac{1-r^n}{1-r}\right]$$

$$\lim_{n\to\infty} S_n = \frac{r}{1-r}$$

i.e $\sum r^n = \frac{r}{1-r}$ $\sum M_n$ [illegible]

Similarily for $\sum r^n . \sin n\theta$, $\sum r^n \cos^2 n\theta$, ... $\sum r^n \sin(a^n\theta)$ all are uniformly cgt for all θ and $0 < r < 1$

② $\sum \frac{x}{n^p + x^2 n^q}$, $x \in [a, b]$, $a > -\infty$; $b < \infty$

Here $\left|\frac{x}{n^p + x^2 n^q}\right| \leq \left|\frac{x}{n^p}\right|$ If $p > 1$ & $q \geq 0$

Let $k = \max\{|a|, |b|\}$

then $\left| \frac{x}{n^p + x^2 n^q} \right| < \frac{k}{n^p} = (M_n)$

$\because \sum \frac{k}{n^p}$ is conv. for $p > 1$

$\Rightarrow \sum \frac{x}{n^p + x^2 n^q}$ is U.C on $[a, b]$ if $p > 1$ & $q \geq 0$

③ $\sum \frac{(-1)^n}{n} |x|^n, \; x \in [-1, 1]$ [Apply Abel's test].

④ $\sum \frac{\cos n\theta}{n^p}$ or $\sum \frac{\sin n\theta}{n^p}$ $p > 0$ $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

⑤ $\sum (\log(n+1))^{-x} \cos nx$, $[a, b]$, $a > 0$, $b < 2\pi$

$v_n(x) = [\log(n+1)]^{-x} = \frac{1}{\log(n+1)^x}$ is Monotonically ↓ ∀ x, & tends to 0 as $n \to \infty$

& S.O.P.S of $\sum \cos nx = \left| \sum_{n=1}^{n} \cos nx \right| < \frac{1}{\sin \frac{x}{2}}$

$$\left| \frac{1}{\sin a/2} \right| = \left(\operatorname{cosec} \frac{a}{2} \right)$$

$\Rightarrow$ S.O.P.S is bdd.

Hence by Dirichlet's Test $\sum [\log(n+1)]^{-x} \cos nx$ is U.C on $[a, b]$, $a > 0$, $b < 2\pi$.

Results

Let $D \leq \mathbb{R}$, $f_n : D \longrightarrow \mathbb{R}$ be seq^n of bdd function and series f_n is U·C to some function f then f is bdd.

let $D \leq \mathbb{R}$, $f_n : D \longrightarrow \mathbb{R}$ be [illegible] cts function and series f_n is U·C to some function f then f is cts.

NEE:
Let $D \leq \mathbb{R}$ is a bdd subset of $\mathbb{R}$, $f_n : D \to \mathbb{R}$ f_n is seq^n of integrable function, series f_n is U·C to some function f then f is integrable function.

if $f_n(x) \xrightarrow{U \cdot C} f$

$$\lim_{n \to \infty} \int_a^b f_n(x) = \int_a^b f$$ integrable limit function,

Let D is compact subset of $\mathbb{R}$, [illegible] $\mathbb{R}$ be a seq n of function on compact set [illegible]. seq^n f_n pointwise [illegible] and some function f is [illegible] or $f_n(x)$ is monotonic then series [illegible] is uniformly cgt.

Functions Of Several Variables :—

As we have Discussed the concept of Single variable Real valued functions and about their Properties, Application etc Now, in this chapter we will mainly concerned with the application of differential calculus to function of more than one-variable.

First, we are going to discuss the cocept of Nbd and limit point of set of n-tupple space ($\mathbb{R}^n$).

The Neighbourhood of a point :-

Let (a,b) be any point of $\mathbb{R}^2$, then the Nbd of point (a,b) is defined as.

The set of values x, y other than (a,b) that satisfy the conditions

$$|x-a| < \delta, \quad |y-b| < \delta$$

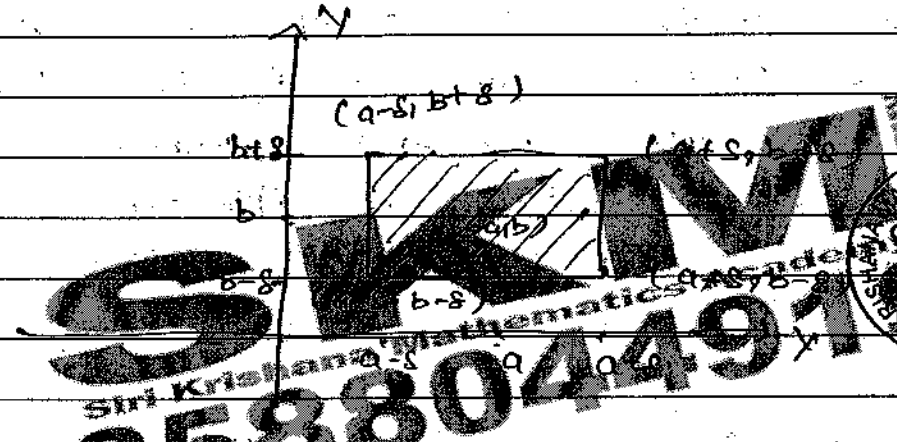
where δ is smaller Arbitary positive No., is said to form a Nbd of the point (a,b).

Thus a Nbd of (a,b) is a square where point on the x-axis moves from $a-\delta$ to $a+\delta$ and on y-axis $b-\delta$ to $b+\delta$.

i.e $(a-\delta, a+\delta \,;\, b-\delta, b+\delta)$

Here x takes any value from $a-\delta$ to $a+\delta$ except a and y from $b-\delta$ to $b+\delta$ except b.

This square $(a-\delta, a+\delta\,;\, b-\delta, b+\delta)$ Near (a,b) can be shown as


Y
$(a-\delta, b+\delta)$
$b+\delta$
$(a+\delta, b+\delta)$
b
(a,b)
$b-\delta$
$(b-\delta)$
$(a+\delta, b-\delta)$
$a-\delta$
a
$a+\delta$
X

Hence the set of $\mathbb{R}^2$; where $x \in (a-\delta, a+\delta)$ & $y \in (b-\delta, b+\delta)$

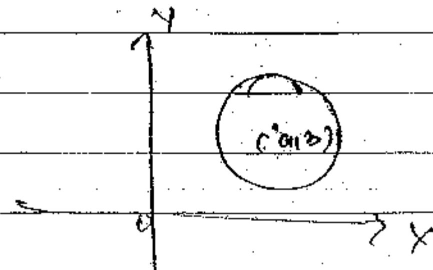
i.e $S = \{(x,y) \in \mathbb{R}^2,\ a-\delta \le x \le a+\delta,\ b-\delta \le y \le b+\delta\}$ is Nbd of any given point of $\mathbb{R}^2$.

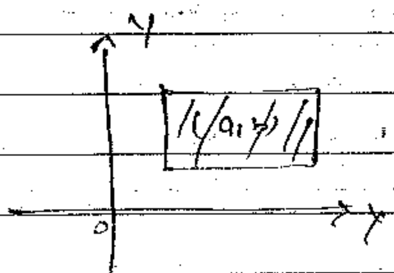
Note :- As we know, for any point $a \in \mathbb{R}$ there are many nbds similarly for any point $(a,b) \in \mathbb{R}^2$, there can be many Nbds.

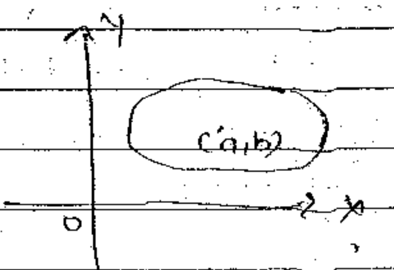
Earlier, we discussed any square $(a-\delta, a+\delta\,;\, b-\delta, b+\delta)$ is Nbd of [illegible] for every [illegible]

Similarly, S [illegible], $(x-a)^2 + (y-b)^2 \le \delta^2\}$ is also a Nbd of point (a,b).

i.e points inside the disc $(x-a)^2 + (y-b)^2 \le \delta^2$ may be taken as Nbd of the point (a,b)







set of points inside disc, Rectangular, ellipse all are different Nbd. of point (a,b)

Limit point of a set (n- variable) :-

Let S be any subset of $\mathbb{R}^3$ and (a,b) be any point of $\mathbb{R}^2$ then (a,b) is said to be limit point of set S if every Nbd of (a,b) contains infinite member of S

Eg:- $S = \left\{\left(\frac{1}{n}, \frac{1}{m}\right); m,n \in \mathbb{N}\right\}$

Component $\frac{1}{n}$ tends to '0' as $n \to \infty$, also component $\frac{1}{m} \to 0$ as $n \to \infty$ Here $\left(\frac{1}{n}, \frac{1}{m}\right) \to (0,0)$ as $m,n \to \infty$, Hence let S has limit point (0,0).

2) $S = [0,1] \times [0,2] \subseteq \mathbb{R}^2$

Here $[0,1] \subseteq \mathbb{R}$ and $[0,2] \subseteq \mathbb{R}$ then the derived set of [0,1] and [0,2] are [0,1] and [0,2] Respectively, then Derived set of $S = S' = [0,1] \times [0,2]$.

3) If $S = (0,1) \times (0,1)$ then
$S' = [0,1] \times [0,1]$.

4) if $S = (0,2) \times \{3\}$ then
$S' = [0,2] \times \{3\}$.

Note :- sets given in Examples (3) & (4) can be

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Shown in two Dimensional plane, then we can easily find the limit points of sets.

Functions of Several Variable :- A function from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called function of several variable, when $m=1$, then called Real valued function of n-variable.

Eg:- $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$f(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_m)$

Here all f_i's are Real Valued n-variable functions

i.e $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$.

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ s.t

$f(x_1, x_2, x_3) = (\underbrace{x_1^2 + x_2^2 + x_3}_{f_1}, \underbrace{\sin x_1 + x_2}_{f_2})$

Here $f_1 : \mathbb{R}^3 \rightarrow \mathbb{R}$ & $f_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$

$f_1(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3)$ & $f_2(x_1, x_2, x_3) = \sin x_1 + x_2$

Eg:- $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, two valued, 2-variable fn.

$f(x_1, x_2) = (f_1, f_2)$

Eg:- $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$; two valued, 3-variable fn.

Eg:- $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, m-valued, n-variable fn.

Domain of Definition: The set of points where function of several variable are going to defined.

Eg:- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t

$$f(x_1, x_2) = \frac{1}{x_1 + x_2}$$

Here if take $x_1 = -x_2$ then funⁿ is Not defined then dod = $\{(x,y) \in \mathbb{R}^2 ;\ x \neq y\}$

Eg :- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f(x_1, x_2) = \left(\log x_1,\ \sin \frac{1}{x_2 - 1}\right)$$

Here $\log x_1$ is Not defined for Negative value of x_1 and $\sin \frac{1}{x_2 - 1}$ is Not defined for $x_2 = 1$.

Hence DOD = $\{(x,y) \in \mathbb{R}^2 :\ x > 0,\ y \neq 1\}$

Eg:- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ s.t

$$f(x, y) = \left(x \cdot \sin\frac{1}{y},\ y \cdot \cos\frac{1}{x},\ x + y\right)$$

f_1 is Not defined for $y = 0$, f_2 is Not defined for $x = 0$, f_3 is defined $\forall\ (x,y) \in \mathbb{R}^2$ Hence

DOD = $\{(x,y) \in \mathbb{R}^2 ;\ x \neq 0,\ y \neq 0\}$ i.e

$\{(x,y) \in \mathbb{R}^2 ;\ x \cdot y \neq 0\}$

Limit of a function of several variable:-

Let S be any subset of $\mathbb{R}^2$ and (a,b) be a limit point of S. f be a function defined on S. then function 'f' is said to a limit [illegible] as a point (x,y) approches close to (a,[illegible] [illegible] ε > 0 ∃ δ > 0 s.t

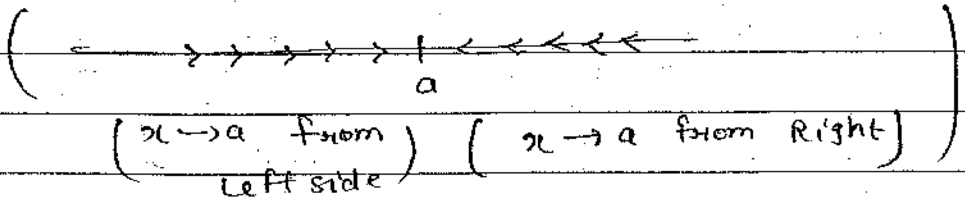
$$|f[illegible] < ε \text{ whenever } |x-a| < δ, |y-b| < δ.$$

In other words a function 'f' tends to a limit 'l' when (x,y) → (a,b) if ∀ ε > 0. ∃ a Nbd of (a,b) s.t $|f(x,y) - l| < ε$
∀ point (x,y) other than (a,b) of the Nbd of (a,b) then we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \quad [\text{symbolically}].$$

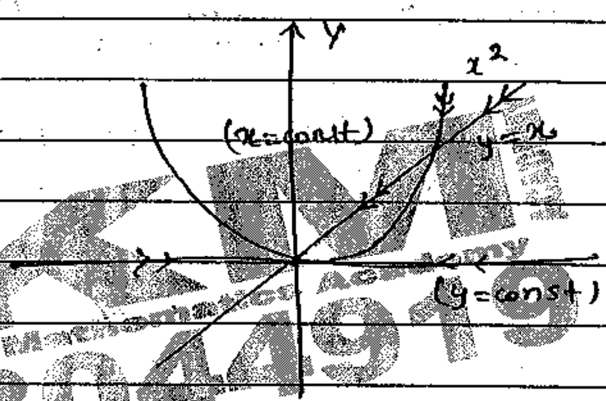
The limit 'l' is called 'double limit' or the 'simultaneous limit' when x, y tend [illegible] simultaneously.

REMEMBER:- As we read [illegible] function of single variable real valued function [illegible] of function 'f' depends on sides, Right hand side (R.H.S) and left hand side (L.H.S) because there only two way to approch close to 'a' R.H.S and L.H.S.



But in several variable there are many path.

Eg:- Let $(0,0) \in \mathbb{R}^2$ then we can approch Near (0,0) through line $y=x$ [in general $y=mx$], through Square function $y=x^2$ [$y=mx^2$] through x-axis and Through y-axis etc.



In single variable if limit from Right and left exist and equal then we write limit exist.
Similarily Here, if limit exist for every possible path and equal then we say limit exist. If there exist two different path limit Not equal then we say limit doesn't exist.

Eg:- $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y) \neq 0 \\ 0 & ; (x,y) \neq 0 \end{cases}$

Now, if we find limit at (0,0) then this limit depend on path,

If we take path $y=x$ then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\cdot x}{x^2+x^2} = \dfrac{1}{2}$

If we take path $y=2x$ then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\cdot 2x}{x^2+4x^2} = \dfrac{2}{5}$

General, $y=mx$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\cdot mx}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$

Hence from different path limit at (0,0) are different So limit doesn't exist at (0,0).

Note:- Wheather a simultaneous limit exist or Not is a Difficult matter, But very often a simple consideration to show limit doesn't exist. i.e we say $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ doesn't exist if ∃ two paths $y=\phi(x)$ and $y=\psi(x)$ along which $(x,y)\to(a,b)$ and $f(x,y)$ tends to different values.

Eg:- $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

if we choose path $y=mx$ [point $(a,b)=(0,0)$].

$$\lim_{(x,y)\to(a,b)} \quad \lim_{x\to 0} f(x,mx) = \frac{x^2-m^2x^2}{x^2+m^2x^2}$$

$$\lim_{(x,y)\to(0,0)} = \frac{1-m^2}{1+m^2} \quad [\text{depending on } m]$$

Hence limit does not exist at (0,0).

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Note:- In particular, generally (not always) choose linear path $y=mx+c$. if point $(a,b)=(0,0)$ then choose $y=mx$. And if along this path limit comes out to be function of m, then we say limit doesn't exist.

Eg:- find limit of $\dfrac{x^3+y^2}{x-y}$ at (0,0)

Put $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x^3 + m^2x^2}{x - mx}$$

limit exist along path $y = mx$ but if put $y = x - mx^3$

then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \frac{x^3 + y^2}{x - y}$ doesn't exist.

over all, limit does Not exist at $(0,0)$.

Note :- If along $y = mx$, limit exist then we say only limit exist along $y = mx$, but we can't say about existence of limit [above Example].

Note :- 1) $\lim\limits_{(x,y)\to(0,0)} \left(\frac{1}{|x|} + \frac{1}{|y|} \right) = \infty$

2) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = 0$

3) $\lim\limits_{(x,y)\to(0,0)} \frac{1}{xy} \sin(x^2y + xy^2) = 0$.

In case simultaneous limit exist [as in (2) & (3)] then from any path we approches close to (a,b) then function will approche at some limit l.

Algo to show 'l' is limit of function of several variable :-

let f be a function of several variable defined on $S \subseteq \mathbb{R}^2$. if $\lim_{(x,y)\to(a,b)} f(x,y)$ exist and equal to 'l'

or if we [illegible] that 'l' is limit or Not then we [illegible] apply the algorithm and can be proved easily by calculating simple steps.

Step→1 simplyfy $|f(x,y) - l|$ to get

$$|f(x,y) - l| \leq \alpha|x-a| + \beta|y-b|$$

or

$$|f(x,y) - l| \leq \gamma|x-a| \cdot |y-b|$$

Step→2 Now set (put or fix).

$\delta = \epsilon$ [for [illegible]

[illegible] $\delta =$ [illegible] [for $\epsilon/2$ form]

Step→3 claim $|f(x,y) - l| < \epsilon$ whenever $|x-a| < \delta$, $|y-b| < \delta$.

Eg:- 1) $f(x,y) = 2x + 3y$, Show that,

$$\lim_{(x,y)\to(1,1)} f(x,y) = 5.$$

Step 1 $|f(x,y) - l|$

$$= |2x+3y-5| = |2x+3y-2-3|$$

$$= |2(x-1)+3(y-1)|$$

$$\leq 2|x-1| + 3|y-1| \quad \text{First form]}.$$

Step-2. set $\delta = \dfrac{\epsilon}{\alpha+\beta}$ (Here $\alpha=2$, $\beta=3$).

$$\delta = \frac{\epsilon}{5}$$

step → 3 $\forall \epsilon > 0$ $\exists$ $\delta = \dfrac{\epsilon}{5}$ s.t

$$|f(x,y) - 5| < \epsilon \quad \text{whenever } |x-1| < \delta,\ |y-1| < \delta.$$

Eg-12 Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ where

$$f(x,y) = \begin{cases} x \sin 1/y + y \cdot \sin 1/x, & x,y \neq 0 \\ 0, & x,y = 0 \end{cases}$$

Soln. --

$$|f(x,y) - l| = \left| x \cdot \sin\frac{1}{y} + y \sin\frac{1}{x} - 0 \right|$$

$$\leq \left| x \sin\frac{1}{y} \right| + \left| y \cdot \sin\frac{1}{x} \right|$$

$$= |x| \cdot \left| \sin\frac{1}{y} \right| + |y| \cdot \sin\left|\frac{1}{x}\right|$$

$$\leq |x| + |y| \quad \left[\left|\sin\frac{1}{x}\right| \leq 1, \ \left|\sin\frac{1}{y}\right| \leq 1 \right]$$

$$\leq 1 \cdot |x-0| + 1 \cdot |y-0|$$

Here $\alpha = 1$, $\beta = 1$ then Put $\delta = \dfrac{\epsilon}{\alpha+\beta} = \dfrac{\epsilon}{2}$

Hence $\forall\ \epsilon > 0\ \exists\ \delta = \frac{\epsilon}{2}$ s.t

$|f(x,y) - 0| < \epsilon$ whenever $|x| < \delta,\ |y| < \delta$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

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Eg-3 Prove that $\lim_{(x,y)\to(0,0)} \frac{x y}{\sqrt{x^2+y^2}} = 0$

Soln :- Let $\forall\ \epsilon > 0$

$|f(x,y) - 0| < \epsilon$

Now $\left| \frac{x \cdot y}{\sqrt{x^2+y^2}} - 0 \right| = \left| \frac{x \cdot y}{\sqrt{x^2+y^2}} \right|$ — (1)

$\because\ y^2 \le x^2+y^2\ \forall\ x \in \mathbb{R}$

$|y| \le \sqrt{x^2+y^2}$

$\frac{|y|}{\sqrt{x^2+y^2}} \le 1$ or $\frac{|x|}{\sqrt{x^2+y^2}} \le 1$

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Now from

$\left| \frac{x \cdot y}{\sqrt{x^2+y^2}} \right| = \frac{|y|}{\sqrt{x^2+y^2}} \cdot |x|$

$\le |x|$ or $|y|$.

$\le 1 \cdot |x-0| + 0 \cdot |y-0|$ | or $0 \cdot |x-0| + 1 \cdot |y-0|$

here $\alpha = 1,\ \beta = 0$. | or here $\beta = 1,\ \alpha = 0$

Set $\delta = \frac{\epsilon}{\alpha+\beta} = \delta = \epsilon$

Hence, $\forall\ \epsilon > 0, \exists\ \delta > 0, \delta = \epsilon$ s.t

$|f(x,y) - 0| < \epsilon$ whenever
$|x-0| < \delta, \ |y-0| < \delta.$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Q Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined as.

$$f(x,y) = \begin{cases} \frac{x^3y - y^3x}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find limit of $f(x,y)$ at $(0,0)$

Soln:- By inspection the function $f(x,y)$, we can say limit may be zero because degree of $f(x,y)$ are 4 and 2 in ~~Number~~ Numerator and denominator Respectively. Now to confirm the limit is zero, we can apply algorithem. if $f(x,y)$ satisfies all the steps with limit zero. Then it will confirm that limit will be zero.

$$|f(x,y) - l| = \left| \frac{x^3y - y^3x}{x^2+y^2} - 0 \right|$$

$$= \left| \frac{xy \cdot (x^2 - y^2)}{x^2+y^2} \right|$$

$= |x \cdot y| \cdot \left| \frac{x^2-y^2}{x^2+y^2} \right|$

$\le |x| \cdot |y| \cdot 1 \qquad \left[\because \left| \frac{x^2-y^2}{x^2+y^2} \right| \le 1 \right)$

$\le 1 \cdot |x-0| \cdot |y-0|$

compare [illegible] $|f(x,y)-0| \le r$ [illegible] $|y-0|$. we have [illegible] Now $= \sqrt{\epsilon}$

$\delta = \sqrt{\epsilon}$.

Hence $\forall\ \epsilon > 0\ \exists\ \delta = \sqrt{\epsilon}$ s.t

$|f(x,y) - 0| < \epsilon$ whenever $|x-0| < \delta,\ |y-0| < \delta$.

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Eg8- $f(x,y) = \begin{cases} \frac{x^3-y^3}{x+y} & x+y \ne 0 \\ [illegible] \end{cases}$

$f(x,y) = \frac{x^3 - [illegible]}{x+y}$

Put $y = -x + mx^3$

$\lim_{x\to 0} \frac{x^3 - (-x+mx^3)}{x - x + mx^3}$

$\lim_{x\to 0} \frac{x^3 \left[1 + \frac{1}{x^2} - m \right]}{mx^3}$

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$$\lim_{x\to 0} \frac{\left[1+\frac{1}{x^2}-m\right]}{m}$$

Limit does Not exist.

Eg:- $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y} ; & x-y \neq 0 \\ 0 ; & x = y \end{cases}$

Put $y = x - mx^3$

$$\lim_{x\to 0} \frac{x^3+(x-mx^3)^3}{x-x+mx^3}$$

$$\lim_{x\to 0} \frac{x^3+x^3-m^3x^9-3xmx^3(x-mx^3)}{x-x+mx^3}$$

$$\lim_{x\to 0} \frac{2x^3-m^3x^9-3x^5m+3x^7m^2}{mx^3}$$

$$\lim_{x\to 0} \frac{2-m^3x^6-3x^2m+3x^4m^2}{m}$$

Depends upon m

$\Rightarrow$ Limit Does Not exist

Eg:- $f(x,y) = \frac{x^2y}{x^2+y^2}$

Sol^n Let $x = r\cos\theta$, $y = r\sin\theta$.
assume 0 is limit

$$|f(x,y)-0| = \left|\frac{x^2y}{x^2+y^2}\right| = \left|\frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)}\right|$$

$= \left| \frac{r^3\cos^2\theta \sin\theta}{r^2 \cdot 1} \right| = |r\cos^2\theta \sin\theta| \leq r < \epsilon$

$\because r^2 = x^2+y^2$

$\Rightarrow \sqrt{x^2+y^2} < \epsilon$

$\Rightarrow x^2+y^2 < \epsilon^2$

$\Rightarrow x^2 < \frac{\epsilon^2}{2}$ [illegible] $y^2 < \frac{\epsilon^2}{2}$

$\Rightarrow$ [illegible] $|y| < \frac{\epsilon}{\sqrt{2}}$

let $\frac{\epsilon}{\sqrt{2}}$ [illegible]

$\Rightarrow |x| < \delta$ & $|y| < \delta$.

$\Rightarrow 0$ is limit.

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Continuity of several variable functions

Let (f) be a function defined on [illegible]. let $(a,b) \in S$ & then (f) is said to [illegible] cts at $x = (a,b)$ [illegible]

$\lim_{(x,y)\to(a,b)} f(x,y)$ [illegible]

i.e limit of $f(x,y)$ at (a,b) = value of function at (a,b)

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Hence, A function f is said to cts at a point (a,b) if $\forall \epsilon > 0\ \exists\ \delta > 0$ s.t

$|f(x,y) - f(a,b)| < \epsilon$ whenever $|x-a| < \delta$, $|y-b| < \delta$.

Q4 $f(x,y) = \begin{cases} \dfrac{x \cdot y}{\sqrt{x^2+y^2}} , & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

Here $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Hence, $f(x,y)$ is cts at $(0,0)$

Q4 $f(x,y) = \begin{cases} \dfrac{1}{|x|} + \dfrac{1}{|y|} ; & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

$f(x,y)$ is Not cts at $(0,0)$ as limit doesn't exist at $(0,0)$.

$\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{1}{|x|} + \dfrac{1}{|y|} \right) = \infty \neq f(0,0)$

Q4 $f(x,y) = \begin{cases} x^2+3y^2 ; & (x,y) \neq (1,0) \ \& \ (0,1) \\ 0 ; & (x,y) = (1,0) \text{ or } (0,1) \end{cases}$

Here $\lim\limits_{(x,y)\to(1,0)} f(x,y) = 1 \neq f(1,0)$

$\lim\limits_{(x,y)\to(0,1)} f(x,y) = 3 \neq f(0,1)$.

Hence $f(x,y)$ is Not cts only at $(1,0)$ & $(0,1)$.

Q4 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ Set

$f(x,y) = \left(x^2+3y^2+5 ; \ x^2+2y+6xy \right)$

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Here f_i are two variable Real value functions w.r.t any ~~cony~~ $(a,b) \in \mathbb{R}^2$

$$\lim_{(x,y)\to(a,b)} f(x,y) = (a^2+3b^2+5,\ a^2+2b+\cos(ab)) = f(a,b).$$

$\Rightarrow f(x,y)$ is cts at every [illegible]

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Result :- [illegible] $F : \mathbb{R}^n \to \mathbb{R}^m$ [illegible]

$$F(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_m)$$

then $\lim_{x\to A} f(x)$ exist iff $\lim_{x\to A} f_i(x)$ exists $\forall$ $i = 1$ to m where $A \in \mathbb{R}^n$ and is limit point.

And, if limit exist $\forall$ i and equal to l (say) then $l = (l_1, l_2, \ldots, l_m)$.

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Eg:- $f : \mathbb{R}^2 \to \mathbb{R}^3$ s.t
$$f(x,y) = (x^2 [illegible]^2, \sin x^2, e [illegible])$$

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Here f_1 is Poly. [illegible] variable x, y ; f_2 is Sine function, f_3 is exponential function. As I know All these function's limit exist at every point of domain of defination. Hence.

$\lim_{(x,y)\to(a,b)} f(x,y)$ exists at every point $(a,b) \in \mathbb{R}^2$

because $\lim_{(x,y)\to(a,b)} f_i(x)$ exist $\forall$ $(a,b) \in \mathbb{R}^2$.

Note :- Polynomials, functions, Sine functions, cosine functions, exponential function and the product of all the function and linear combination of product of these function are always continuous at every point of Domain of defination and if $f(x) = (f_1, f_2, \ldots, f_n)$ are given and f_i's are of such type of function then we can say directly $f(x)$ is throught cts.

NET J-2011

Que $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t $f(x, y) = (3x - 2y + x^2, 4x + 5y + y^2)$

Here $f_1(x, y) = 3x - 2y + x^2$ and $f_2(x, y) = 4x + 5y + y^2$ both are Poly. (2-variable) function. So f_1 & f_2 both will cts at every point Hence $f(x, y)$ is cts at every point of $\mathbb{R}^3$.

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NET J-2012

Que $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t $f(x, y) = (7x + x^4, 3x + 4y + y^4)$

$f(x, y)$ is cts at every point of $\mathbb{R}^2$.

Que $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ s.t

$$f(x, y) = \begin{cases} \left(\dfrac{(xy)}{\sqrt{x^2+y^2}} ; \dfrac{x \cdot y}{x^2+y^2} ; x^2+y^2 \right) & ; (x,y) \neq 0 \\ (0, 0, 0) & ; (x, y) = 0 \end{cases}$$

$\lim\limits_{(x,y) \to (0,0)} \dfrac{x \cdot y}{x^2+y^2}$ doesn't exist (by Putting $y = mx$)

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does Not exist at $(0,0)$.

$\Rightarrow$ $f(x,y)$ is Not cts at $(0,0)$.

Note :- Let f be a continuous at point (a,b) of its Domain of Defination then $f(x,b)$ is cts function of x at $x=a$ and $f(a,y)$ that of y at $y=b$, But converse Not true.

Eg:-

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

Investigate at $(0,0)$

$$f(x,0) = \frac{x^2-0}{x^2+0} = \frac{x^2}{x^2} = 1$$

Hence $\lim_{x\to 0} f(x,0) = 1$ = value of $f(x,0)$ at $x=0$

Similarily $\lim_{y\to 0} f(0,y) = -1$ = value of $f(0,y)$ at $y=0$

Hence $f(x,0)$ & $f(a,y)$ both are cts but $f(x,y)$ is Not cts as limit does Not exist at $(0,0)$.

Algebra of limit :-

If 'f' and 'g' are two functions on some Nbd of a point (a,b) such that $\lim_{(x,y)\to(a,b)} f(x,y) = l$ and

$\lim_{(x,y)\to(a,b)} g(x,y) = m$ then.

1) $\lim(f+g) = \lim f + \lim g$

2) $\lim(f-g) = \lim f - \lim g$

3) $\lim(f\cdot g) = \lim f \cdot \lim g$

4) $\lim(f/g) = \dfrac{\lim f}{\lim g}$ [provided $m \neq 0$].

$$= \frac{l}{m}$$

Eg:- $\lim\limits_{(x,y)\to(2,1)} \dfrac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$

let $xy-2 = t$.

$3xy-6 = 3t$.

Now $t \to 0$ as $(x,y) \to (2,1)$

$\lim\limits_{t\to 0} \dfrac{\sin^{-1}t}{\tan^{-1}3t} = \left(\dfrac{0}{0} \text{ form}\right)$

$\Rightarrow \lim\limits_{t\to 0} \dfrac{1/\sqrt{1-t^2}}{3\cdot 1/1+9t^2} = \dfrac{1}{3}$

Hence, $\lim\limits_{(x,y)\to(2,1)} f(x,y) = \dfrac{1}{3}$

Eg:- $f(x,y) = \begin{cases} x\sin 1/y + y\sin 1/x & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Let 0 is the limit of $f(x,y)$

Put $y = mx$.

$|f(x,y) - f(a,b)|$

$\lim\limits_{x\to 0} f(x,mx) = \lim\limits_{x\to 0} x\cdot\sin\dfrac{1}{mx} + mx\sin\dfrac{1}{x}$

$= 0+0 = 0$

$$\left| x\sin\frac{1}{y} + y\sin\frac{1}{x} \right| \le |x|\left|\sin\frac{1}{y}\right| + |y|\left|\sin\frac{1}{x}\right|$$

$$\le |x|+|y| < \epsilon$$

$\Rightarrow |x| < \epsilon/2$, $|y| < \epsilon/2$.

$\Rightarrow$ 0 is limit of $f(x,y)$

$\Rightarrow$ f is cts

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Eg :- $f(x,y) = \begin{cases} \sin x/y & ; y \neq 0 \\ 0 & ; y = 0 \end{cases}$

check function is cts at $(1,0)$

Put $y = mx$.

$$\lim_{\substack{x\to 1\\ y\to 0}} \sin\frac{x}{mx} = \lim_{x\to 1} \sin\frac{1}{m}$$

limit depends only m

$\Rightarrow$ limit does not exist.

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Directional Derivative :- (D.D)

Let S be a subset of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$ be a function defined on S with values in $\mathbb{R}^m$. Then the directional derivative of 'f' at 'c' in the direction of 'u' denoted by $f'(c,u)$ and is defined by

$$f'(c,u) = \lim_{h\to 0} \frac{f(c+hu)-f(c)}{h}$$

Note :- Some require that $\|u\|=1$ (unit vector) but It is Not assumed Here because As we know In every Direction there is always a unit vector If u is any vector of $\mathbb{R}^n$ then unit vector in the direction of u can be calculated as $\frac{u}{\|u\|}$. Hence direction of u and $\frac{u}{\|u\|}$ are same.

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Meaning of $f(c+hu)$:- Let $f: S \to \mathbb{R}^m$ $(S \subseteq \mathbb{R}^n)$ be a function with value in $\mathbb{R}^n$. $f(c+hu)$ means, how 'f' changes as we move from a point c in S along a line segment to a nearby point $c+u$, $(u \neq 0)$. Each point on the segment can be expressed as $c+hu$ (where h is Real). u describe the direction of the line segment And c is an interior point of S.

Note :- if $u=0$ then $f'(c,0) = \lim_{h\to 0} \frac{f(c)-f(c)}{h} = 0$ $\forall$ c in S

Que :- Let $f(x,y) = \begin{cases} x+y & ; \text{ if } x=0 \text{ or } y=0 \\ 1 & ; \text{ else} \end{cases}$

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Find the directional derivative at $(0,0)$ is the direction of u_1, u_2, u_3 where

$u_1 = (a_1, a_2)$ $a_1 \neq 0$, $a_2 \neq 0$

$u_2 = (a, 0)$ $a \neq 0$

$u_3 = (0, a)$ $a \neq 0$

Soln :- (I) $c = (0,0)$, $u_1 = u = (a_1, a_2)$; $a_1, a_2 \neq 0$

$$\lim_{h\to 0} \frac{f(c+hu)-f(c)}{h} = \lim_{h\to 0} \frac{f[(0,0)+h(a_1,a_2)]-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(ha_1, ha_2) - 0}{h}$$

$$f'(c,u_1) = \lim_{h\to 0} \frac{1-0}{h} = \infty \quad \left[\begin{matrix} ha_1 \neq 0 \ \& \\ ha_2 \neq 0 \end{matrix}\right].$$

II $\quad c = (0,0), \quad u_2 = u = (a,0), \quad a \neq 0$

$$\lim_{h\to 0} \frac{f(c+hu) - f(c)}{h} = \lim_{h\to 0} \frac{f(ha,0) - f(0,0)}{h}$$

$$f'(c,u_2) = \lim_{h\to 0} \frac{ah}{h} = a.$$

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~~III) $c = (0,0), \quad u_2 = (a,0) \quad a \neq 0.$~~

~~$c = (0,0) = u_2 = u = (a,0), \quad a \neq 0$~~

~~$\lim_{h\to 0} \frac{f(c+hu) - f(c)}{h} = \lim_{h\to 0} \frac{f(ha,0) - f(0,0)}{h}$~~

~~$f'(c,u_2) = \lim_{h\to 0} \frac{ah}{h} = a$~~

III $\quad c = (0,0), \quad u_3 = (0,a) \quad a \neq 0$

$$\lim_{h\to 0} \frac{f(c+hu) - f(c)}{h} = \lim_{h\to 0} \frac{f(0,ha) - f(0,0)}{h} = a = f'(c,u_3)$$

Hence $f'(c,u_1)$ doesn't exist and $f'(c,u_2) = f'(c,u_3)$
$= a$, both exist.

Qus $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t $f(x) = \|x\|^2$; $x = (a,b)$

Hence $f(x)$ can be written as $f(x) = x\cdot x = \|x\|^2$

Let c be any point of $\mathbb{R}^2$ and u be any vector the directional derivative of 'f' at c in the direction of u is

$$\lim_{h\to 0} \frac{f(c+hu)-f(c)}{h} = \lim_{h\to 0} \frac{(c+hu)\cdot(c+hu)-c\cdot c}{h}$$

$$= \lim_{h\to 0} \frac{c\cdot c+2h\, c\cdot u+h^2 u\cdot u-c\cdot c}{h}$$

$$= \lim_{h\to 0} \frac{\|c\|^2+2h\, cu+h^2\|u\|^2-\|c\|^2}{h}$$

$$= \lim_{h\to 0} (2cu+h\cdot\|u\|^2)$$

$f'(c,u) = 2\cdot cu\cdot$

If $c=(0,0)$; $u=(a_1,a_2)$ then $f'(c,u)=0$

if $c=(1,0)$, $u=(0,1)$ then $f'(c,u) = 2(1,0)\cdot(0,1)$
$= 0$

if $c=(1,1)$, $u=(1,0)$ then $f'(c,u) = 2(1,1)\cdot(1,0)$
$= 2.$

Meaning of Partial Derivative :-

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (f should be Real valued function) $(1,0)$, $(0,1)$ are the unit vector corresponding to co-ordinate axis x and y. the $\frac{\partial f}{\partial x}$ means the directional derivative in direction of $(1,0)$ i.e in the direction of x-axis.

Hence $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $c=(a,b)$ are the point $c=(a,b)$ are the directional derivative in the direction of $(1,0)$, $(0,1)$ Respectively [in the direction of x and y - axis Repect].

In General:- if $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathbb{R}$-valued function, the Partial derivative $\frac{\partial f}{\partial x_k}$ at the point 'c' is D.D. in the direction of u_k where u_k is the unit vector the vect direction of k^{th} - axis.

[i.e $u_k = (0,0,0, - - - - 1, 0, 0, - - -)$]
(kth Position)

i.e $f'(c,u_k)$ is the partial Derivative of f in the Direction of u_k

i.e $f'(c,u_k) = \frac{\partial f}{\partial x_k}$ at c.

Note:- $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t
$f(x_1, x_2, - - -, x_n) = (f_1, f_2, f_3, - - -, f_m)$
Then $f'(c,u)$ exist iff $f_i'(c,u)$ exist $\forall\, i = 1$ to m.

Note:- The existence of directional Derivative Doesn't ensure the Continuity

Eg:- $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Let $c = (0,0)$ & $u = (1,0)$

then $f'(c,u) = \lim_{h\to 0} \frac{f[(0,0)+h(1,0)] - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{f(h,0) - 0}{h} = 0$

And $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{x\to 0} \frac{1}{1+m^2}$ [putting $x = my^2$]

Hence $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{1+m^2}$ doesn't exist [Path Dependent]

Here Directional Derivative exist but 'f' is Not cts even limit doesn't exist at $(0,0)$.

Result 2:- A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t $f(x) = (f_1, f_2, \ldots, f_m)$ and if $f(x)$ satisfies the condition $f(\alpha\cdot x + \beta\cdot y) = \alpha\cdot f(x) + \beta f(y)$. [Linear function] $\forall\ x, y \in \mathbb{R}^n$ & $\forall\ \alpha, \beta$ scalars the directional derivative of 'f' at 'c' in the direction of u is given by $f'(c,u) = f(u)$.

i.e if 'f' is linear map then

$\lim_{h\to 0} \frac{f(c+hu) - f(c)}{h} = \lim_{h\to 0} \frac{f(c) + h\cdot f(u) - f(c)}{h}$

$= \lim_{h\to 0} f(u)$

$$\boxed{f'(c,u) = f(u)}$$

Eg. $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t $f(x,y) = (x,-y)$

This f satisfy Linear Property. then Directional Derivative at (a,b) in the direction of u.

$$f'(c,u) = f(u) \quad \forall\ c \in \mathbb{R}^n.$$

if $u=(1,0)$ then $f'(c,u) = f(u) = f(1,0) = (1,0)$
if $u=(0,1)$ then $f'(c,u) = f(u) = f(0,1) = (0,-1)$
if $u=(2,3)$ then $f'(c,u) = f(u) = f(2,3) = (2,-3)$

Differentiability :-

In $\mathbb{R}$:- Let f is defined on $S \subseteq \mathbb{R}$ then f is said to be differentiable at $a \in S \cap S'$ if $\exists$ α depending on f and $x=a$ both s.t

$$f(a+h) - f(a) - \alpha h = h\,\phi(h)$$

$$\text{and } \lim_{h\to 0} \phi(h) = 0$$

and α become the derivative of f at $x=a$.

$$\lim_{h\to 0}\left(f(a+h) - f(a) - \alpha h\right) = \lim_{h\to 0} h\,\phi(h)$$

$$\because \lim_{h\to 0} \phi(h) = 0 \quad \text{for differentiability}$$

$$\Rightarrow \lim_{h\to 0}\left(f(a+h) - f(a)\right) = \lim_{h\to 0} \alpha . h.$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \alpha \quad \left[\text{derivative of 'f' at } x=a\right].$$

$$\Rightarrow \boxed{f'(a) = \alpha} \quad [\text{defination in } \mathbb{R}].$$

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The idea of differentiability in $\mathbb{R}$ can be extended on $\mathbb{R}^2$ and is defined as –

'f' is said to be differentiable at a point (a,b) if $\exists\ \alpha, \beta$ depending on f and (a,b) both whenever.

$$f(a+h, b+k) - f(a,b) - \alpha h - \beta k = h\phi_1(h,k) + k\phi_2(h,k)$$

First Form.

Or

$$f(a+h, b+k) - f(a,b) - \alpha h - \beta k = \sqrt{h^2+k^2}\cdot\phi(h,k).$$

Second Form

then $\lim\limits_{(h,k)\to(0,0)} \phi_i(h,k) = 0 \quad \forall\ i$

Note if $\lim\limits_{(h,k)\to(0,0)} \phi_i(h,k) \neq 0$ for any i then f is Not differentiable.

Note :- α, β used in the defination are the directional derivative of f at (a,b) in the direction of $(1,0)$ and $(0,1)$ Respectively.

i.e $\alpha = \lim\limits_{h\to 0} \dfrac{f[(a,b)+h(1,0)] - f(a,b)}{h}$

$$\boxed{\alpha = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = \frac{\partial f}{\partial x}(a,b)}$$

Similarily

$$\boxed{\beta = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = \frac{\partial f}{\partial y}(a,b)}$$

Que Find α, β for the following function at $(0,0)$.

$$f(x,y) = \begin{cases} x \cdot y \cdot \sin 1/x & ; \ x \neq (a,0) \quad a \neq 0 \\ 0 & ; \ x = (0,0) \ \text{or else.} \end{cases}$$

Solⁿ :- $\alpha = \lim\limits_{h \to 0} \dfrac{f(a+h, b) - f(a,b)}{h}$ $[(a,b) = (0,0)]$

$= \lim\limits_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0-0}{h} = 0$

$\beta = \lim\limits_{k \to 0} \dfrac{f(0, 0+k) - f(0,0)}{k} = \lim\limits_{k \to 0} \dfrac{0-0}{k} = 0$

Hence $\boxed{\alpha = \beta = 0}$

For Differentiability :-

$f(a+h, b+k) - f(a,b) - \alpha \cdot h - \beta \cdot k = f(0+h, 0+k) - f(0,0) - 0 - 0$

$= f(h,k)$

$= h \cdot k \cdot \sin \frac{1}{h} + 0$

$= h \cdot \left(k \cdot \sin \frac{1}{h}\right) + k \cdot 0$

$= h\,\phi_1(h,k) + k\,\phi_2(h,k)$

[First form].

Now, $\lim\limits_{(h,k) \to (0,0)} \phi_1(h,k) = \lim\limits_{(h,k) \to (0,0)} k \cdot \sin \frac{1}{h} = 0$

and $\lim\limits_{(h,k) \to (0,0)} \phi_2(h,k) = \lim\limits_{(h,k) \to (0,0)} 0 = 0$

Hence $f(x,y)$ is diff at $(0,0)$

Question :- check differentiabity at $(0,0)$ for

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Soln :-

$$\alpha = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\beta = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Now, $f(a+h, b+k) - f(a,b) - \alpha \cdot h - \beta \cdot k =$

$$f(h,k) - f(0,0) - 0\cdot h - 0\cdot k$$

$$= \frac{hk}{\sqrt{h^2+k^2}}$$

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$$= \frac{\sqrt{h^2+k^2} \times h\cdot k}{\sqrt{h^2+k^2}\,\sqrt{h^2+k^2}}$$

$$= \sqrt{h^2+k^2} \cdot \frac{hK}{h^2+k^2}$$ [second form]

Here $\phi(h,k) = \dfrac{hk}{h^2+k^2}$

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{h\cdot k}{h^2+k^2} = \frac{m}{1+m^2}$$ [Put $k = mh$]

Hence limit of $\phi(h,k)$ doesn't exist because
It's path dependent

So $f(x,y)$ is Not diff at $(0,0)$

Note :- Existense of directional derivative at (a,b) doesn't es ensure differentiability there at.

Eg:- In above question α, β ~~d~~ both exist and is 0 but f is Not differentiable.

Hence, Existence of partial (directional) derivative at (a,b) only Necessary condition not sufficient.

i.e.

$$\boxed{\begin{array}{l} f \text{ is diff at } (a,b) \Rightarrow \alpha, \beta \text{ both exist.} \\ \text{If } \alpha, \beta \text{ does Not exist} \Rightarrow f \text{ is Not diff.} \end{array}}$$

NET J-2011

Que The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$; defined by

$$f(x,y) = (\underbrace{3x-2y+x^2}_{f_1}, \underbrace{4x+5y+y^2}_{f_2}) \text{ then.}$$

a) f is discontinuous at (0,0)

[✓ b)] f is cts at (0,0) and all directional derivative exist at (0,0).

c) f is diff at (0,0) but derivative Df(0,0) is Not invertible.

[✓ d)] f is diff at (0,0) and derivative Df(0,0) is invertible.

Sol^n :- f_1, f_2 are Poly. function so both are cts.
So f(x,y) is cts at every point so at (0,0)

Now directional derivative at (0,0) in the direction of $(a_1, a_2) \in \mathbb{R}^2$ [In all direction].

$$\lim_{h\to 0}\frac{f(c+hu)-f(c)}{h} = \lim_{h\to 0}\frac{f[(0,0)+h(a_1,a_2)]-f(0,0)}{h}$$

$$= \lim_{h\to 0} f[(ha_1, ha_2)]-(0,0)$$

$$= \lim_{h\to 0}\frac{(3a_1h-2ha_2+h^2a_1^2,\ 4a_1h+5a_2h+h^2a_2^2)}{h}$$

$$= \lim_{h\to 0}(3a_1-2a_2+ha_1^2,\ 4a_1+5a_2+ha_2^2)$$

$$= (3a_1-2a_2,\ 4a_1+5a_2)$$

Hence directional derivative exist at $c=(0,0)$ in the Direction of $u=(a_1,a_2)$.

i.e $f'(c,u) = (3a_1-2a_2,\ 4a_1+5a_2)$.

Now $\alpha = f'(c,u) = f'[(0,0),(1,0)]$ $\left[\begin{matrix} c=(0,0) \\ u=(1,0)\end{matrix}\right]$ — (i)

Here $(a_1,a_2)=(1,0)$ put in (i)

$\alpha = (3,4)$ & $\beta = (-2,5)$ [By Putting $(a_1,a_2)=(0,1)$]

$$f(0+h,0+k)-f(0,0)-\alpha h-\beta k = (3h-2k+h^2,\ 4h+5k+k^2)$$
$$-(0,0)-h(3,4)-k(2,5)$$
$$= (h^2,k^2)$$
$$= (h^2,0)+(0,k^2)$$
$$= h(h,0)+k\cdot(0,k)$$

First form

$\phi_1(h,k)=(h,0)$ & $\phi_2(h,k)=(0,k)$

& $\lim_{(h,k)\to(0,0)} \phi_i(h,k)=0\ \forall\ i=1,2$

Hence f(x,y) is diff at (0,0)

Now $Df(x_1, x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \end{bmatrix} = \begin{bmatrix} 3+2x_1 & -2 \\ 4 & 5+2x_2 \end{bmatrix}$

$Df(0,0) = \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 15+8 = 23 \neq 0$

$\Rightarrow$ Df(0,0) is invertiable.

Eg: $f(x) = |xy|^{1/2}$ at $(0,0)$

$f(0+h, 0+k) - f(0,0) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k)$

$A = f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}$

$B = f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = 0$

$|hk|^{1/2} = 0 + 0 + \sqrt{h^2+k^2}\,\phi(h,k)$

$\frac{|hk|^{1/2}}{\sqrt{h^2+k^2}} = \phi(h,k)$

$\Rightarrow \phi(h,k) = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$

$\lim_{(h,k) \to (0,0)} = \lim_{(h,k) \to (0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$

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$h = r\cos\theta, \quad k = r\sin\theta$

$$\lim_{\theta\to 0} \frac{|r\sin\theta \cdot r\cos\theta|^{1/2}}{(r^2(\cos^2\theta+\sin^2\theta))^{1/2}} = \lim_{\theta\to 0} |\sin\theta\cos\theta|^{1/2}$$

depends upon θ

$\Rightarrow$ limit does Not [illegible]

$\Rightarrow$ f is Not diff[illegible]

Eg:- $f(x,y) = |xy|$ at $(0,0)$.

$$A = f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = 0$$

$$B = f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = 0$$

$$f(0+h, 0+k) - f(0,0) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k)$$

$$|hk| = 0 + 0 + \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\phi(h,k) = \frac{|hk|}{\sqrt{h^2+k^2}}$$

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}}$$

Put $h = k$

$$\lim_{h\to 0} \frac{|h^2|}{\sqrt{2h^2}} = \lim_{h\to 0} \frac{h}{\sqrt{2}} = 0$$

f is diff at $(0,0)$

Eg: $f(x,y) = |xy|^{1/3}$

$$\lim_{(h,k)\to(0,0)} = \lim_{r\to0} \frac{|r\cos\theta . r\sin\theta|^{1/3}}{r}$$

$$= \lim_{r\to0} \frac{r^{2/3}|\sin\theta\cos\theta|}{r}$$

$$\lim_{r\to0} \frac{1}{r^{1/3}}|\sin\theta\cos\theta|$$ does not exist

$\Rightarrow$ f is Not diff.

If Let (a,b) be the point of Domain of function and f_x exist at (a,b) and f_y is ctc at (a,b)
$\Rightarrow$ f is diff at (a,b).

Ist order Partial Derivative at (a,b)

$$f_x(a,b) = \lim_{h\to0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_{xy}(a,b) = \lim_{k\to0} f(a,b+k) - f(a,b)$$

$$f_{xx}(a,b) = \lim_{h\to0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

$$f_{yy}(a,b) = \lim_{k\to0} \frac{f_y(a,b+k) - f_y(a,b)}{k}$$

$$f_{xy}(a,b) = \lim_{h\to0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$f_{yx}(a,b) = \lim_{k \to 0} \frac{f_x(a,b+k) - f(a,b)}{k}$$

Jacobian Matrix or Matrix of Derivative.

Let S be a subset of $\mathbb{R}^n$ and $f : S \longrightarrow \mathbb{R}^m$ is a function s.t $f(x_1, x_2, \dots, x_n) = (f_1, f_2, \dots, f_m)$ then the Jacobian Matrix is denoted by $Df(x)$. Here $x = (x_1, x_2, \dots, x_n)$ and calculated as.

$$Df(x) = Df(x_1, x_2, \dots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

(or)

$$Df(x) = \begin{bmatrix} D_1 f_1(x) & D_2 f_1(x) & \cdots & D_n f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) & \cdots & D_n f_2(x) \\ \vdots & & & \\ D_1 f_m(x) & D_2 f_m(x) & & D_n f_m(x) \end{bmatrix}_{m \times n}$$

Jacobian Matrix at a the point $x = c$ i.e

$$Df(c) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(c) & \frac{\partial f_2}{\partial x_2}(c) & \cdots & \frac{\partial f_1}{\partial x_n}(c) \\ \frac{\partial f_2}{\partial x_1}(c) & \frac{\partial f_2}{\partial x_2}(c) & \cdots & \frac{\partial f_2}{\partial x_n}(c) \\ \vdots & \vdots & & \\ \frac{\partial f_m}{\partial x_1}(c) & \frac{\partial f_m}{\partial x_2}(c) & \cdots & \frac{\partial f_m}{\partial x_n}(c) \end{bmatrix}_{m \times n}$$

If determinant of $Df(c)$ is Non-zero (if $m=n$). then we say $Df(c)$ is invertible.

Note:- As we know total derivative at point c can be calculated by $f'(c)$. and total derivative at the point c in the direction of vector u_k (k^{th} axis unit vector) denoted by $T(u_k)$ and defined as

$$\boxed{T(u_k) = f'(c; u_k)}$$

Note:- Let f be a linear Mapping then the derivative $f'(c)$ exist $\forall$ $c \in \mathbb{R}^n$.

Result:- Let f be a Linear then $f'(c)$ exist $\forall$ $c \in \mathbb{R}^n$ and, The Jacobian Matrix and Matrix Representation of Linear Mapping with usual Basis will be same i.e

$$\boxed{Df(c) = [f; B, B]}$$

As we know Every Matrix is Linear Mapping. Here $Df(x)$ is Jacobian Matrix so It will be Linear Mapping then this f will be equal to Linear map f. Hence we can written as.

$$\boxed{Df(x,y) = f} \quad \forall \ (x,y) \in \text{Domain}$$

NET
D-2011

Question:- $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, s.t $L(x,y) = (\underset{f_1}{x}, \underset{f_2}{-y})$ is

a) Diff everywhere on $\mathbb{R}^2$

b) Diff. only at $(0,0)$

c) $DL(0,0) = L$

d) $DL(x,y) = L \;\; \forall \; (x,y) \in \mathbb{R}^2$.

Sol^n Let $C = (a,b) \in \mathbb{R}^2$.

$$\alpha = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{(a+h,-b) - (a,-b)}{h}$$

$$= \lim_{h\to 0} \frac{(h,0)}{h} = (1,0)$$

Similarily $\beta = (0,-1)$

~~Now~~

$$f(a+h, b+k) - f(a,b) - \alpha h - \beta \cdot k$$

$$= (a+h, -b-k) - (a,-b) - h(1,0) - k(0,-1)$$

$$= (0,0)$$

$$= h(0,0) + k(0,0)$$

Here $\phi_1(h,k) = \phi_2(h,k) = (0,0)$

Hence $\lim\limits_{(h,k)\to(0,0)} \phi_i(h,k) = (0,0)$

Hence, $f(x,y)$ is diff $\forall \; (a,b) \in \mathbb{R}^2$

Now.

$$Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \;\; \forall \; (x,y) \in \mathbb{R}^2$$

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∵ $Df(x,y)$ is Mapping (Transformation) and can be written as.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x, -y)$$

Hence $Df(x,y)(x,y) = (x,-y)$ $\forall (x,y) \in \mathbb{R}^2$

i.e. $Df(x,y) = h$ $\forall (x,y) \in \mathbb{R}^2$

∵ $Df(x,y) = h$ $\forall (x,y) \in \mathbb{R}^2$ So,

$$\boxed{Df(0,0) = h}$$

Note:- In above question function is Linear then use Result, that if (f) is Linear then f is Diff and $Df(x,y) = f$ $\forall (x,y) \in \mathbb{R}^2$. Hence directly tick your answer.

NETD-2011

Que Let $f(r,\theta) = (r\cos\theta, r\sin\theta)$ with $r \neq 0$ which of the following Statement are correct.

a) The Linear Transformation $Df(r,\theta)$ is Not zero for any $(r,\theta) \in \mathbb{R}^2$ with $r \neq 0$

b) f is 1-1 on $\{(r,\theta) \in \mathbb{R}^2, r \neq 0\}$

c) for any (r,θ), $r \neq 0$, f is 1-1 on Nbd of (r,θ)

d) $Df(r,\theta) = r^2 I$ for any $(r,\theta) \in \mathbb{R}^2$; $r \neq 0$

soln :- $Df(r,\theta) = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

as we know for any (r,θ) $r \neq 0$ $\cos\theta$ & $\sin\theta$ will Not be zero i.e for any θ, $\sin\theta$ and $\cos\theta$ are Not zero together.
Hence for some θ, if $\sin\theta = 0$ then $\cos\theta$ will Not be zero, ~~for~~ Similar for some θ if $\cos\theta = 0$ then $\sin\theta$ will Not be zero Hence $Df(r,\theta)$ is Not zero for (r,θ) with $r \neq 0$.

Now let $(r,\theta_1) = (r, \pi/4)$ & $(r,\theta_2) = (r, \frac{\pi}{4} + 2\pi)$

$f(r,\theta_1) = f(r,\theta_2) = \left(r\cdot\frac{1}{\sqrt{2}}, r\cdot\frac{1}{\sqrt{2}}\right)$

i.e for two value (r,θ_1) and (r,θ_2), f has same value $\Rightarrow$ f is Not 1-1 on $\{(r,\theta) \in \mathbb{R}^2, r \neq 0\}$.

If we take a Nbd of (r,θ) in which θ and $\theta + 2\pi$ type elements doesn't belong at a time.
Because $\sin\theta$ and $\cos\theta$ will given same value at θ and $\theta+2\pi$ {2π is Period of $\sin\theta$ & $\cos\theta$}. Hence f is 1-1 on such type Nbd of (r,θ).
In the last option they are saying $Df(r,\theta) = r^2 I$

i.e $Df(r,\theta) = r^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$

But $Df(r,\theta) \neq r^2 I$
Hence option (a) & (c) are correct.

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Q14 If $f: S \to S$ is a function then we denote $f^k = fofof\cdots of$ (k times). Let f_1 and f_2 are functions on $\mathbb{R}^2$.

$f_1(x,y) = (x+1, y+3)$

$f_2(x,y) = (x-3, y-2)$ then.

a) For any $k \in \mathbb{N}$, ∃ a unique $(a,b) \in \mathbb{R}^2$ s.t. $f_1^k(0,0) = f_2^k(a,b)$

b) For any $k \in \mathbb{R}$ and any $k \in \mathbb{N}$, there is atmost one soln y for $f_1^k(0,0) = f_2^k(x,y)$.

c) ∃ $(a,b) \in \mathbb{R}^2$ s.t. $f_1^k(a,b) \neq f_2^k(x,y)$ for any $(x,y) \in \mathbb{R}^2$

d) f_1 is a linear transformation.

Soln: $f_1 o f_1 = f_1[f_1(x,y)] = f_1(x+1, y+3) = (x+2, y+6)$

$f_1 o f_1 o f_1 = f_1[f_1 o f_1] = f_1(x+2, y+6) = (x+3, y+9)$

Hence in general $f_1 o f_1 o \cdots o f_1$ (k times) $= (x+k, y+3k)$

i.e. $\boxed{f_1^k(x,y) = (x+k, y+3k)}$

similarly $\boxed{f^k(x,y) = (x-3k, y-2k)}$

For option (a) $f_1^k(0,0) = f_2^k(a,b)$

$(k, 3k) = (a-3k, b-2k)$

$\Rightarrow a-3k = k \Rightarrow a = 4k$

& $b-2k = 3k \Rightarrow b = 5k$.

Hence we can say for any $k \in N$ ∃ $(a,b) = (4k, 5k)$ s.t. $f_1^k(0,0) = f_2^k(a,b)$

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for option (b) $f_1^k(0,0) = f_2^k(x,y)$

$(k,k) = (x-3k,\ y-2k)$

$x = 4k \ \& \ y = 3k.$

$\because$ $k \in \mathbb{N}$ so $x \in \mathbb{N}$ Hence if $x = 4k$ then there is always unique soln y s.t. $f_1^k(0,0) = f_2^k(x,y)$. But they are saying $x \in \mathbb{R}$ so for Negative x ∄ any y s.t. $f_1^k(0,0) = f_2^k(x,y)$ [Same for $x \neq 4k$].

Hence we can say $\forall x \in \mathbb{R}$ & $\forall k \in \mathbb{N}$: There is atmost one Soln y s.t. $f_1^k(0,0) = f_2^k(x,y)$.

for option (c) Let $(x,y) = (1,2) \in \mathbb{R}^2$

$f_1^k(a,b) = f_2^k(1,2)$

$(a+k,\ b+3k) = (1-3k,\ 2-2k)$

$\Rightarrow a+k = 1-3k \Rightarrow a = 1+4k$

$b+3k = 2-2k \Rightarrow b = 2-5k.$

Hence if we take $(x,y) = (1,2)$ then ∃ $(a,b) = (1-4k,\ 2-5k)$ s.t. $f_1^k(a,b) = f_2^k(1,2)$

Similarly we can show $\forall (x,y) \in \mathbb{R}^2$ ∃ $(a,b) \in \mathbb{R}^2$ s.t. $f_1^k(a,b) = f_2^k(x,y)$, So option c is incorrect.

For Option (d) f_1 is Not linear as $f(0,0) \neq (0,0)$

D-2013
NET
Q4)

Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a diff function. Let $Df(x)$ be the Derivative of 'f' at $x \in \mathbb{R}^m$ then,

a) $Df(0)(u) = 0 \ \forall \ u \in \mathbb{R}^m$

b) $Df(x)(u)=0 \ \forall \ u \in \mathbb{R}^m$ and Some $x \in \mathbb{R}^m$ only if 'f' is a constant.

c) $Df(x)(u)=0 \ \forall \ x \ \& \ u \in \mathbb{R}^m$ only if 'f' is constant.

d) if 'f' is Not a const function, then $Df(x)$ is a one to one function four some $x \in \mathbb{R}^m$.

Soln:- As we know $Df(x)$ is Linear Map and Let $Df(x)=T$ and if f is const. then $Df(x)=0 \ \forall \ x \in \mathbb{R}^m$ i.e. then $T=0$ (zero Linear operator) then $T(u)=0$ $\forall \ u \in \mathbb{R}^m$

Hence $T(u)=0 \Rightarrow Df(x)(u)=0 \ \forall \ x \in \mathbb{R}^n \ \forall \ u \in \mathbb{R}^m$

option (c) is correct.

Now four other options:- Let $f(x,y) = (xy, 0)$.

$$Df(x,y) = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$$

Now $Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow Df(0,0)(u)=0 \ \forall \ u \in \mathbb{R}^2$ and for some $(0,0)$ but $f(x,y)$ is Not const. $\Rightarrow$ option 2 is discard.

Again $f(x,y)=(x \cdot y, 0)$ is Not constant function. but $Df(x,y) = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$. is Not 1-1 as $Df(x,y)$ is singular. i.e $Df(x,y)$ is Not Invertible so (d) is incorrect.

Now let $f(x,y) = (x,y)$

$$Df(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \forall \ (x,y) \in \mathbb{R}^2$$

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$Df(0,0)(u) \neq 0 \;\forall\, u \in \mathbb{R}^2$

Here $Df(0,0)$ is invertible so $Df(0,0)(u)$ will be zero only for $u=(0,0)$ Not $\forall\, u \in \mathbb{R}^2$.
Hence, option (a) is Not correct.

NET J-2013
(Q.4) Let $z = x+iy$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x,y) = f(z) = z^2 = (x^2-y^2, 2xy) \in \mathbb{R}^2$. Let $Df(a)$ denote the derivative at (a) then.

a) $(Df)(a)\cdot h = 2ah$ Here $a = a_1+ia_2$ & $h = h_1+ih_2$

b) $(Df)(a) = 2\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$, $a = (a_1,a_2) \in \mathbb{R}^2$.

c) f is one to one on $\mathbb{R}^2$

d) For any $a \in \mathbb{R}^2 - \{(0,0)\}$, f is one to one on some Nbd of (a).

Soln $a = a_1+ia_2$, $h = h_1+ih_2$

$$a\cdot h = (a_1+ia_2)(h_1+ih_2)$$
$$(a_1h_1-a_2h_2)+i(a_1h_2+a_2h_1)$$

i.e $a\cdot h = (a_1h_1-a_2h_2,\ a_1h_2+a_2h_1)$ [2-tupple form] $\longrightarrow (1)$

$$Df(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$Df(a) = Df(a_1,a_2) = 2\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}$$

(b) is correct

$$Df(a)h = 2\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= 2(a_1h_1 - a_2h_2,\ a_2h_1 + a_1h_2)$$

$Df(a).h = 2ah$. [using (i)]

option (a) is correct.

Now let $(x_1,y_1) = (1,1)$ and $(x_2,y_2) = (-1,-1)$

$f(x_1,y_1) = f(x_2,y_2) = (0,0)$

f is Not one-one on $\mathbb{R}^2$

option (c) is Not correct.

And if we take smaller Nbd of (1,1) or of (-1,-1) function will 1-1. otherwise we can say that $a \in \mathbb{R}^2 - (0,0)$ and take Nbd of a which doesn't contain (0,0) then on that Nbd fun. will be 1-1. Hence (d) is correct.

(iii) check continuity, diff at (0,0)

$$f(x,y) = \begin{cases} x\cdot\sin\frac{1}{x} + y\cdot\sin\frac{1}{x} & ;\ x, y \neq 0 \\ x\cdot\sin\frac{1}{x} & ;\ x \neq 0, y = 0 \\ y\sin\frac{1}{y} & ;\ x = 0, y \neq 0 \\ 0 & ;\ x = 0, y = 0 \end{cases}$$

Here $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(x,y)$ $\Rightarrow$ f is cts at (0,0)

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$$\alpha = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h \sin 1/h}{h} = \lim_{h\to 0} \sin 1/h$$

does Not Exist. Similarly β does Not exist. Hence f is Not diff at $(0,0)$.

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Ex $f(x,y) = |x| + |y|$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \Rightarrow f \text{ is cts at } (0,0)$$

Now $\alpha = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h} \begin{cases} 1 & \text{from RHS} \\ -1 & \text{'' L.H.S} \end{cases}$

Here α does Not exist

$\Rightarrow f(x,y)$ is Not diff at $(0,0)$.

Young's Thm:- If f_x and f_y are both diff. at a point (a,b) of Domain of defination of a function f, then

$$f_{xy} \text{ at } (a,b) = f_{yx} \text{ at } (a,b)$$

Differentiability of f_x and f_y says that they exist in a certain Nbd of (a,b) and all the second order Partial Derivatives f_{xx}, f_{xy}, f_{yx}, f_{yy} exist at (a,b)

Schwarz's Thm:- If f_y exist in some Nbd of (a,b) and f_{yx} is cts at (a,b) then f_{xy} exist and equal to f_{yx}.

$\boxed{\text{or}}$

If f_x exist in some nbd of (a,b) and f_{xy} is cts at (a,b) then f_{yx} exist and equal to f_{xy} at (a,b).

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Note :- Diff of f_x and f_y in Young's thm are only sufficient condition Not Necessary.

Note :- Existence of f_x (or f_y) and continuity of f_{xy} (or f_{yx}) are only sufficient Not Necessary.

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Q.11 Let $f: [0,1] \longrightarrow \mathbb{R}$ be defined by $f(x)$

$$= \begin{cases} x\cos\frac{\pi}{2x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ then.

a) f is cts on $[0,1]$
b) f is diff on $(0,1)$ and its derivative is bdd on $(0,1)$.
c) f is of Bdd variation on $[0,1]$.
d) f is R.I on $[0,1]$.

Q-12 Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a cts function; which of the following is always true.

(a) $f^{-1}(U)$ is open for all open set $U \subset \mathbb{R}$
(b) $f^{-1}(C)$ is closed for all closed set $C \subseteq \mathbb{R}$
(c) $f^{-1}(K)$ is compact for all compact sets $K \subseteq \mathbb{R}$
d) $f^{-1}(G)$ is connected for all connected sets $G \subseteq \mathbb{R}$.

Q-13 Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be diff with $0 < f'(x) < 1$ $\forall$ x, then

a) f is increasing and bdd.
b) f is increasing and f is Reimann integrable on $\mathbb{R}$
c) f is increasing and f is U.C
d) f is of Bounded variation.

Q-14 Let $f: [0,1] \longrightarrow (0,\infty)$ be two continuous fn, suppose $f(0) = 1$ and $f(1) = 7$ then
a) f is Uniformly continuous and is Not onto

b) f is increasing and $f[0,1] = [1,7]$

c) f is Not uniformly continuous.

d) f is Not bdd

5) Let $f : [a,b] \to [c,d]$ be a Montonic and bijective, then

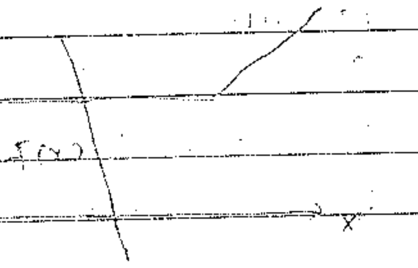
a) f is cts but f^{-1} Need be

b) f and f^{-1} both are cts.

c) If $b-a > d-c$, then f is decreasing function

d) f is Not uniformely cts.

6) The graph of function f is given below



then the derivative of f.

a) Exist Every where except at $x=1$.

b) Does Not Exist for anywhere

c) Exists only at $x=1$

d) Exist Every where and is cts.

7) $f(x) = \begin{cases} x^2 \sin 1/x^2 ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$ on $\mathbb{R}$ then

a) f is Diff. Everywhere and with continuity derivative.

b) f is diff Everywhere other than the point 0.

c) f is Nowhere Differentiable

d) f is diff but It's derivative Not cts.

⑧ Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x^2]$, the points of discontinuity of f are.

a) only the integer points

b) all Rational No.

c) $\{\sqrt{n}\ ;\ n$ is Non-Negative integer$\}$.

d) all Real Number.

⑨ The function $f(x) = |x| + 3$ is

a) cts as well as diff on $\mathbb{R}$

b) cts on $\mathbb{R}$ but Not diff anywhere on $\mathbb{R}$

c) cts on $\mathbb{R}$ but ~~Not~~ diff any at all but a finitely many points on $\mathbb{R}$.

d) ~~$f : \mathbb{R}$~~ Not cts on $\mathbb{R}$.

⑩ $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \begin{cases} 0 & ; x \text{ is Not a +ve integer} \\ 1/n & ; x \text{ is +ve integer } n. \end{cases}$

then.

a) cts everywhere b) Cts Nowhere

c) cts at 0 only d) cts at every except at positive integers.

① A Monotonic function.

a) is always cts.
b) can be Nowhere cts.
c) is cts only if has Intermediate value Property
d) can be discts at infinitely many points

② Let $f(x) = |\sin \pi x|$, $x \in \mathbb{R}$ then

a) f is Nowhere cts
b) f is diff everywhere.
c) f is cts everywhere but diff Nowhere
d) f is cts everywhere and diff everywhere except at integral values of x.

⑬ $f : [0,1] \to \mathbb{R}$ be discts Monotonic function, the Range of f is

a) a Bounded set that is Not interval.
b) a Bounded interval
c) an unbdd connected set
d) a set that is Neither bdd Nor connected.

⑭ If f is diff on $\mathbb{R}$ s.t $f(0)=5$ and $f(x)<5$ for $x \neq 0$ then
a) $f'(0)=5$ b) $f'(0)=-5$ c) $f'(0)=0$ d) $f'(0)=25$

⑮ Function $f(x) = |x^2|, x>0$ is

[✓ a)] cts but Not U.C

b) U.C but Not cts.

c) Neither cts Nor U.C

d) Discontinuous everywhere

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⑯ Which of the following two functions has exactly two points of Discontinuity.

a) $f(x) = \begin{cases} 1 ; & x>0 \\ 0 ; & x \le 0 \end{cases}$

[✓ b)] $f(x) = \begin{cases} 1 ; & 0 \le x \le 1 \\ 0 ; & \text{otherwise} \end{cases}$

c) $f(x) = \begin{cases} x , & 0 \le x \le 1 \\ 0 ; & \text{otherwise} \end{cases}$

[✓ d)] $f(x) = \begin{cases} 1 ; & x \in Q \\ 0 ; & x \notin Q \end{cases}$

⑰ $f(x) = 1-|1-x|$ on $\mathbb{R}$ is

a) Not cts

b) cts but diff.

c) Diff but Not cts

[✓ d)] Diff only at one point.

⑱ $f : [0,1] \rightarrow \mathbb{R}$ is a fun., which of the following is possible.

a) f is bdd but Not cts

b) f is unbdd and cts

c) f is diff. but its derivative is Not cts

d) f is cts but Not integrable

(19) Let f and g be two convex function then.

a) $f \cdot g$ is convex c) $f-g$ is convex
b) $f+g$ is convex d) f/g is convex.

(20) Let $f, g : \mathbb{R} \to \mathbb{R}$ are two U·C function with $g(x) > 0 \ \forall x \in \mathbb{R}$. Then which of following is true.

(i) $f \cdot g$ is U·C for all f&g.
(ii) $f+g$ is U·C for all f&g.
(iii) f/g is U·C for all f&g
(iv) $f \cdot g$ is Never continuous.

(21) $\lim_{x \to 1} \frac{x^3-1}{x-1}$ is

a) ∞ b) 0 c) 1 d) 3

(22) Let $f : [a,b] \to \mathbb{R}$ be a cts function and $f(a) < f(b)$ then by intermediate value thm.

a) $f([a,b]) = [f(a), f(b)]$ (b) $f([a,b]) \supseteq [f(a), f(b)]$
c) $f([a,b]) \subseteq [f(a), f(b)]$ d) $f([a,b]) \neq [f(a), f(b)]$.

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(23) An Example of a function on the Real line $\mathbb{R}$ that is cts but Not uniformly cts.

a) Constant function c) $\sin x$
b) Identity " d) x^2.

㉔ Let $f: \mathbb{R} \to \mathbb{R}$ be diff. which of these will follow form mean value thm.

a) $\forall\ a, b \in \mathbb{R}$ if $a < c < b$ then $\frac{f(b)-f(a)}{b-a} = f'(c)$

b) for some a, b in $\mathbb{R}$ and $\forall\ c \in (a,b)$

$\frac{f(a)-f(b)}{b-a} = f'(c)$

ⓒ for all $a, b \in \mathbb{R}$ $\exists$ c in (a,b) s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$

ⓓ $\forall\ a, b \in \mathbb{R}$ there is unique element c in (a,b) s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$.

㉕ The function $f(x) = \begin{cases} x \sin 1/x \ ; & x \neq 0 \\ 0 \ ; & x = 0 \end{cases}$ is

a) diff at 0 but Not cts
b) has 2nd derivative at origin.
c) cts at origin but Not diff.
d) Neither cts, Nor diff at the origin.

㉖ Suppose f is Real valued function on $[-1,1]$ s.t $|f|$ is cts. which of the following is true

a) f is cts. b) f Need Not be cts.
c) f is Not cts d) f is bdd.

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Q(27) which of the following functions is Not U.C in $[2,\infty)$

a) $\exp(x)$ b) $\frac{1}{x}$ c) $\frac{1}{x^2}$ d) $\sin x$.

Q-128 which of the following statements is true.

a) All cts function from $\mathbb{R}$ to $\mathbb{R}$ are U.C.
b) All cts function from $\mathbb{R}$ to $[0,1]$ are U.C.
c) All cts function from $[0,1]$ to $\mathbb{R}$ are U.C.
d) All cts function from $[0,\infty)$ to $\mathbb{R}$ are U.C.

Q-129 Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $A=\{x\in\mathbb{R}; x\geq 0\}$ $B=\{x\in\mathbb{R}; x<0\}$ then

a) f is cts. $\Rightarrow$ f/A and f/B cts, converse is false.
b) f/A & f/B cts $\Rightarrow$ f is cts converse is Not true.
c) f is cts $\Leftrightarrow$ f/A & f/B are cts.
d) No Relation b/w continuity of f and f/A and f/B.

Q-130 if the differentiable on $[a,b]$ and $f(a)=f(b)=0$ then.

a) $f\left(\frac{a+b}{2}\right)=0$ (b) $f'(\xi)=0$ for some $a<\xi<b$.

c) $f(x)=0 \; \forall \; x \in [a,b]$ (d) $f(x)=0$ for some $x\in(a,b)$

Q-131 Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x)=\begin{cases} x & \text{if } x\in\mathbb{Q} \\ -x & \text{if } x\in\mathbb{Q}^c \end{cases}$

then

a) f is cts Nowhere. (b) f is cts exactly at one point
c) f is cts everywhere (d) f is cts only at irrational point

Q-132 The equation $\sin x = \{x ; x \in \mathbb{R}\}$ has

a) No Root in $[-2,2]$ (b) All Roots in $[-2,2]$.
c) exactly one Root in $[-2,2]$ and Some Root outside $[-2,2]$
d) Infinitely many roots in $[-2,2]$.

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Q-133 Let $f : \mathbb{R} \to \mathbb{R}$ be diff. every where then

a) F is strictly increasing $\Rightarrow F'(x) = 0 \ \forall x$, but converse is fail.
b) $f'(x) > 0 \ \forall x \Rightarrow$ f is strictly increasing.
c) $f'(x) > 0 \ \forall x \Leftrightarrow$ f is strictly increasing.
d) There is No Relation between positivity of $f'(x)$ and Monotonocity of f.

Q-134 Which of the following is Not differentiable at $x = 2$.

(a) $|2x|$ (b) $2|x-1|$ (c) $|x^2-4|$ (d) $|2+x|$

Q-135 if a function is such that $f(x) = x$ at $x = 2$ and $F(x) = 1$ at $x \neq 2$ then

1) $F(x)$ is cts 2) $\lim_{x \to \infty} F(x) = -\infty$ and $\lim_{x \to -\infty} F(x) = \infty$

c) $\lim_{x \to \infty}$ Doesn't exist d) $F(x)$ is discts.

Q→36 The equation $3\sin x = \frac{x}{3}$ has Alumber of solution.

a) 0, (b) 1 c) finitely many
d) Infinite.

Q→37 let $p(x) = x^n + 5x^2 + 7x$, n odd, $n \geq 3$ fix $\alpha \in \mathbb{R}$.
$S = \{x \in \mathbb{R}$ s.t $p(x) = \alpha\}$ then.

a) S is empty b) S is Infinite and Non empty
c) S is c'bly infinite d) S is unc'ble.

Q→38 let $f : \mathbb{N} \to \mathbb{R}$ be any function s.t f takes value in $\mathbb{Z}$ assume that $\lim_{n\to\infty} f(n)$ exist.
Then.

a) f is constant.
b) $\exists\ l, m \in \mathbb{Z}$, $l \neq m$ s.t $f^{-1}(\{m\})$ & $f^{-1}(\{l\})$ are infinite.
c) $f^{-1}(\mathbb{N})$ and $f^{-1}(\{\cdots -n, \cdots, -2, -1\})$ are infinite.
d) $\exists\ K \in \mathbb{N}$ s.t $\lim_{n\to\infty} f(n) = f(K)$

Q→39 Let $f : [0,1] \to \mathbb{R}$ be a cts function. then

a) f maps a bounded seq^n to cauchy seq^n.
b) f maps an unbdd seq^n to a convergent seq^n.

c) f maps Monotonic seq^n to convergent seq^n.

d) f maps bdd seq^n to bdd seq^n.

(40) Using the fact that $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$, $\sum_{1}^{\infty} \frac{(-1)^n}{n(n+1)}$ equals:

a) $1 - 2\log 2$ b) $1 + \log 2$ c) $(\log 2)^2$

d) $-(\log 2)^2$

(41) Let $I = \{1\} \cup \{2\} \subset \mathbb{R}$ [illegible] $\in \mathbb{R}$ Let $\phi(x) = \text{dist}(x, I) = \text{Inf}$ [illegible] $; y \in I\}$ then

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a) ϕ is cts on $\mathbb{R}$ but Not Diff only at $x = 1, 3/2$ and 2.

b) ϕ is cts on $\mathbb{R}$ but Not Diff only at $x = 1$.

c) ϕ is cts on $\mathbb{R}$ but Not Diff only ~~on~~ at $x = 1 \,\&\, 2$

d) ϕ is discts somewhere on $\mathbb{R}$.

(42) Consider the function $f(x) = e^{-x}$ and its Taylor approximation $g(x)$ of degree 3. For $x = 1/3$ $g(x)$ is

a) Positive and less than 1

b) Negative and less than -2

c) Positive and greater than 1

d) less than 1 but greater than ·75.

(43) Let $F = \{ f : \mathbb{R} \longrightarrow \mathbb{R} ;\ |f(x) - f(y)| \leq K|(x-y)|^{\alpha} \}$ $\forall\, x, y \in \mathbb{R}$ and for some $\alpha > 0$ which of the following is / are true:

a) Every $f \in F$ is cts

b) Every $f \in F$ is uniformely cts.

c) Every Differentiable function f is in F.

d) Every $f \in F$ is differentiable.

Q-144 $f(x) = |\cos x| + |\sin(2-x)|$, At which of the following points (f) is Not Differentiable.

a) $\left\{ (2n+1)\frac{\pi}{2} , n \in Z \right\}$

b) $\{ n\pi ; n \in Z \}$

c) $\{ n\pi + 2 , n \in Z \}$.

d) $\left\{ \frac{n\pi}{2} , n \in Z \right\}$

Q-145 Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions defined by,

$$f(x) = 1 + x - [x] \quad \text{and} \quad g(x) = \begin{cases} -1 ; & x < 0 \\ 0 ; & x = 0 \\ 1 ; & x > 0 \end{cases}$$

where $[x]$ is the greatest integer less than or equal to x. Then $(f \circ g)(x)$ is equal to

a) x (b) $f(x)$ (c) $g(x)$ (d) 1.

Q-147 If f is a cts function from S to $\mathbb{R}$ where $S \subseteq \mathbb{R}$ then which of the following is Not possible.

a) S is bdd but f(S) is bdd
b) S is bdd but f(S) is unbdd
c) S is open but f(S) is closed
d) S is closed but f(S) is open.

Q-148 choose the correct statements

a) f(x) is Polynomial in x and $f'(x) \neq 0$ $\forall$ $x \in [a,b]$ then f(x) does Not have any Root in [a,b].

b) f(x) is Polynomial in x and $f'(x) \neq 0$ $\forall x \in [a,b]$ then f(x) can have atleast one Root in [a,b].

c) if f(x) is Poly. in x and $f'(x) \neq 0$ $\forall$ $x \in [a,b]$ then f(x) can have atmost one Root in [a,b].

d) let $a_n = 1+\frac{1}{5}+\frac{1}{9}+ - - +\frac{1}{4n-3}$ then $\langle a_n \rangle$ does Not cgt.

Q-149 let f(x) be defined on [-2,3] is given by

$$f(x) = \begin{cases} -1 ; & -2 \leq x < 0 \\ x-1 ; & 0 < x < 2 \end{cases} \quad \text{and } g(x) = f(|x|) + |f(x)|.$$

Then g(x) is equal to.

a) $\begin{cases} -x ; & -2 \leq x < 0 \\ 0 ; & 0 \leq x < 1 \\ x-1 ; & 1 \leq x \leq 2 \end{cases}$ b) $\begin{cases} -x , & -2 \leq x < 0 \\ x-1 ; & 0 < x < 2 \end{cases}$

c) $\begin{cases} -x ; & -2 \le x < 0 \\ 0 ; & 0 \le x < 1 \\ 2(x-1) ; & 1 \le x \le 2 \end{cases}$ d) None of these.

50 If f be a cts function defined by f(x)

$= \begin{cases} ax+b ; & x < -1 \\ x^2+1 ; & -1 \le x \le 1 \\ ax+b ; & x > 1 \end{cases}$

Here a, b are Real Numbers. then Product a.b is equal to :

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a) 2 b) -4 c) -2 d) 0

(51) If $f : [0,1] \to \mathbb{R}$ is cts maps and has only irrational values, then

a) F is unbdd Function.
b) f Need not to be constant map.
c) f must be constant map.
d) such map doesn't exist.

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(52) Which of the following statements is always true?

a) if f is cts is a Domain D then f attains its bound atleast once in D.

b) if f is defined on [a,b] s.t. f(a) ≠ f(b) and 'f' assumes every value b/w $f(a)$ & $f(b)$ atleast once then f is cts.

c) The image of closed interval under a cts function may be an open interval.

d) None of these.

53 Let $f : [a,b] \to [c,d]$ be a Monotonic and bijective function then.

a) f is cts but f^{-1} Need Not be
b) f Need Not be cts.
c) f and f^{-1} both are cts.
d) f is Not U.C

54 $f(x) = [x] + x \sin\frac{1}{x}$ then in $[-2\pi, 2\pi]$.

a) $f(x)$ is discts but limit exist at every point.
b) $f(x)$ is cts.
c) $f(x)$ is Monotonic
d) limit does Not exist at some point of interval $[-2\pi, 2\pi]$.

55 Let $f(x) = \begin{cases} x, & 0 \le x < \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$ then

a) The Rolle's thm applicable and $\exists$ c, $0 < c < 1$ s.t $f'(c) = 0$

b) The Rolle's theorem is Not applicable because $f(0) \neq f(1)$.

c) The Rolle's Thm is not applicable because f is not cts on $[0,1]$.

d) The Rolle's thm is not applicable because f is not diff. on $(0,1)$.

56 If $a+b+c=0$ then quadratic equation $3ax^2+2bx+c=0$ has

a) ~~Root~~ No Roots $\in [0,1]$
b) atleast one Root $\in [0,1]$
c) two Roots $\in [0,1]$
d) None of these.

57 Let $f(x) = 2\sin x$ then $f(x)$ has

a) one fixed point in $(-\pi,\pi)$
b) No fixed point in $\left(\frac{-\pi}{2},\pi\right)$
c) Three fixed point in $(-\pi,\pi)$.
d) Four fixed point in $(0,\pi)$.

58 If $f(x) = \begin{cases} x^2+ax+1 & ; x \in \mathbb{Q} \\ ax^2+2x+b & ; x \in \mathbb{Q}^c \end{cases}$ is cts at $x=1$ & $x=e$. then

a) $a = 1 - \frac{1}{e}$, $b = e$
b) $a = 0$, $b = 1 - \frac{1}{e}$
c) $a = 1 - \frac{1}{e}$, $b = 0$
d) $a = e$, $b = 1 - \frac{1}{e}$

(59) f(x) and g(x) are cts function s.t $f(1) \neq g(1)$ and $f(-1) \neq g(-1)$ then

$$\phi(x) = \lim_{n\to\infty} \frac{x^n \cdot f(x) + g(x)}{x^n+1}$$

a) is cts on IR
b) has one point of discontinuity
c) Not diff.
d) has two point of discontinuity.

(60) Let $f : D \longrightarrow \mathbb{R}$; $D \subseteq R$ is a function for which set $\left\{ \frac{f(x)-f(y)}{x-y} ;\ \forall\ x, y \in D \ \&\ x \neq y \right\}$ is bdd ; then

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a) f is constant function b) f is cts function
b) f is uniformely cts fun d) F is Diff on D.

(61) If $2a+3b+6c=0$, then atleast one Root of the equation $ax^2+bx+c=0$ has been assured in the interval.

a) (0,1) (b) (1,2) (C) (2,3) d) Not Root can be assured.

(62) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ is diff Function, suppose $f'(x) > f(x)\ \forall\ x \in \mathbb{R}$ and $f(\ ) = 0$ then
a) $\exists\ c \in (1,\infty)$ s.t $f(c)=0$
b) $f(x) > 0,\ \forall\ x \in (1,\infty)$
c) $f(x) < 0,\ \forall\ x \in (1,\infty)$

d) f(x) has countable Number of zeros.

(63) choose the correct statements.

a) set of points of Removable discontinuity of a function f may be uncible.

b) set of point of Removable discontinuity of a function f may be cible.

c) Let $f(x) = |x^3|$ then $f'''(0)$ doesn't exist.

d) All above statements are correct.

(64) Pick out correct statements.

a) If $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$ if f is cts of some points α of $\mathbb{R}$ then f is diff everywhere.

b) If $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$ if f is cts at some point $\alpha \in \mathbb{R}$ then f is U.C on $\mathbb{R}$.

c) If $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$ if f is cts at some point $\alpha \in \mathbb{R}$ then f is bdd function on $\mathbb{R}$.

d) All of above are correct

(65) If f(x) is a Poly function satisfying

$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(3) = 28$ then

$f(4) = ?$

(a) 65 (b) [illegible] (c) [illegible] (d) None of these

(66) Let $f(x) = [x] - \left[\frac{x}{3}\right]$, then

a) $\lim_{x\to 3} f(x)$ doesn't exist

b) $f(x)$ is not cts at $x=3$

c) $\lim_{x\to c} f(x) = 2$

d) $f(x)$ is cts at $x=3$.

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(67) Choose the correct statements

a) $f(x) = \frac{1}{1+x^2}$ is not uniformly cts on $\mathbb{R}$.

b) $\exists$ f and g which both are bdd and uniformly cts on $A \subseteq \mathbb{R}$ but $f \cdot g$ is not u.c on A.

c) Every u.c function on a bdd subset A of $\mathbb{R}$ is bounded on A.

d) $f(x) = \sin\frac{1}{x}$ is not uniformly cts on $(0,\infty)$.

(68) Let $f : \mathbb{R} \to \mathbb{R}$ s.t

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{if } x<0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{if } x>0 \\ c & \text{if } x=0 \end{cases}$$

then f is discontinuity at $x=0$ if

(a) $a = -3/2,\ b = 1,\ c = 1/2$ (b) $a = -3/2,\ b = 0,\ c = 1/2$

(c) $a = 1/2,\ b = 1,\ c = -3/2$ (d) $a = -3/2,\ b = 1,\ c = -1/2$

(10) Which of the following is/are true

a) $\log \frac{x+y}{2} \ge \frac{\log x + \log y}{2} \quad \forall\ x, y > 0$

b) $e^{x+y/2} \le \frac{e^x + e^y}{2} \quad \forall\ x, y > 0$

c) $\sin \frac{x+y}{2} \le \frac{\sin x + \sin y}{2} \quad \forall\ x, y > 0$

d) $\frac{(x+y)^k}{2^k} \le \max\{x^k, y^k\} \quad \forall\ x, y > 0\ \&\ \forall\ k \ge 1.$

(11) For $x \in \mathbb{R}$, let $f(x) = \int_1^{2x} \frac{1}{1+t^2}\,dt$ then derivative of f at 1, $f'(1)$ is equal to.

(a) 0 (b) $1/2$ (c) $5/2$ (d) $2/5$.

(12) In which of Ranges given below does the inequality $1 - \cos(x^3) \le \frac{x^6}{2}$ holds?

(a) $\forall\ x \in (-\infty, \infty)$ (b) $-\pi/2 \le x^2 \le$ [illegible]

(c) $\pi/2 \le x^2 \le 3\pi/2$ (d) None of these

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(13) Let $f(x) = x^2$ on $[0,1]$. For which value of δ given below is it true that if $|x-y| < \delta$ then $|f(x) - f(y)| < \left(\frac{1}{\sqrt{2}}\right)^{[illegible]}$?

(a) $\delta = \left(\frac{1}{\sqrt{2}}\right)^{1/101}$ (b) $\delta = \left(\frac{1}{\sqrt{2}}\right)^{1/102}$

(c) $S = \left(\frac{1}{\sqrt{2}}\right)^{1/103}$ (d) None of these.

(74) Let $f : (0,1) \to \mathbb{R}$ be a 3 times Diff Function s.t $f(1/2) = f'(1/2) = f''(1/2) = f'''(1/2) = 0$ and $f'''(x) = 0$ $\forall$ $x \in (0,1)$. Then which of the following statements is false

a) No such function Exists.
b) The function f', f'', f''' Coincide more than one point in the interval $(0,1)$
c) Any such function is diff. idefinitely.
d) Any such function is uniformly cts.

(75) Let $f : [0,1] \to [0,1]$ be cts. Let f be diff on $(0,1)$ and $|f'(x)| \leq \frac{1}{2}$, $x \in (0,1)$, which of the following statements is true.

(a) $\exists$ a unique $c \in [0,1]$ s.t $f(c) = c$
(b) $\exists$ a unique $c \in (0,1)$ s.t $f(c) = c$
(c) There can be two distinct points c & d in $[0,1]$ s.t $f(c) = c$ and $f(d) = d$.
(d) There Need Not exist c in $[0,1]$ s.t $f(c) = c$.

76 Let $f(t) = |t| + |t-1|$, $t \in [-2,2]$ and

$$F(x) = \int_{-2}^{x} f(t)\,dt, \quad x \in [-2,2]$$

, which of the Following statement is true.

a) f is twice Diff on (-2,2)

b) f is diff. but Not twice Diff.

c) f is Not diff at 0 and 1.

d) f is cts, But Not diff on (-2,2)

Q4 Which of the following Real valued functions f defined on IR have the property f is cts and $f\circ f = f$

(a) $f(x) = \begin{cases} |x| ; & x \in [-1,1] \\ x^2 ; & x \notin [-1,1] \end{cases}$

(b) $f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 1 ; & x \geq 1 \\ 0 ; & x \leq 0 \end{cases}$

(c) $f(x) = \begin{cases} x ; & x \in [-1,1] \\ 1 ; & x \geq 1 \\ -1 ; & x \leq -1 \end{cases}$

(d) $f(x) = \begin{cases} 1 ; & x \in [-23, 27] \\ 22+x ; & x \leq -23 \\ -26+x ; & x \geq 27. \end{cases}$